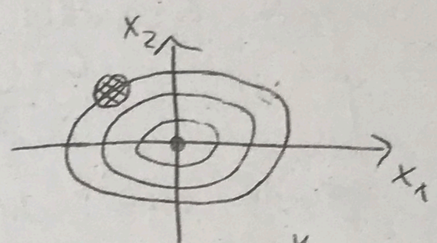
Lecture 2

L2 16.04.19

(1)

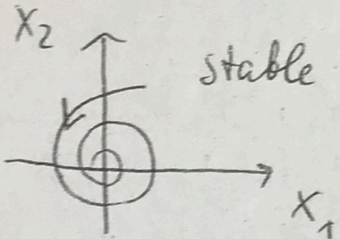
Phase portrait $\to$ phase space with phase trajectories

periodicity
- closed trajectories
- phase volume $\Delta V = const$

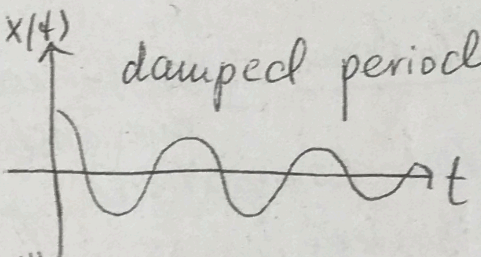


"center"

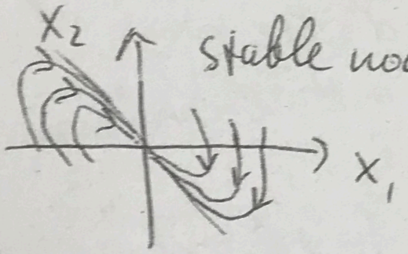
dissipative systems
- for $t \to \infty$ $\Delta V \to 0$



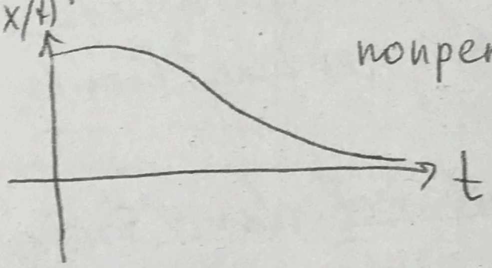
stable focus



damped periodic process (decaying oscillations)



stable node



nonperiodic motion

Dyn. systems (classification)

- continuous (flow)
- discrete-time (maps)

- linear
- nonlinear

- autonomous $\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$ $\to$ parameters

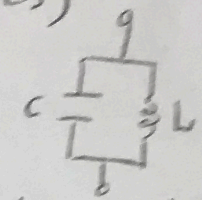
(the evolution law does not explicitly depend on time; no external forcing)

- nonautonomous $\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu}) + \xi(t)$

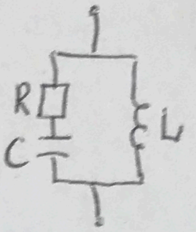
noise or periodic external forcing

(2)

- conservative (called Hamiltonian in mechanics)



- dissipative



(energy decreases in time due to friction or dissipation)

phase volume element is time-dependent

special case: systems with negative dissipation (ΔV increases)

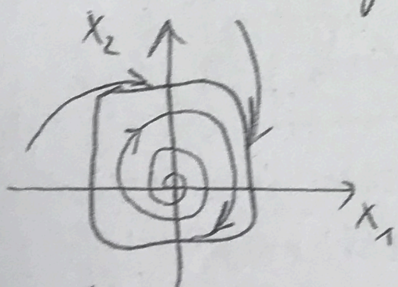
systems with positive dissipation (ΔV decreases)

Self-sustained oscillations : nonlinear and nonconservative

the basic characteristics (amplitude, frequency, waveform) are determined by system parameters and are independent, within certain limits, of initial conditions

Limit cycle : an isolated closed trajectory in the phase space

in the vicinity of the limit cycle there are no more closed orbits!



stable limit cycle

Examples:

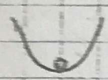
- beating of a heart
- daily rhythms in human body temperature and hormone secretion
- self-excited in bridges and airplane wings

Fixed points (steady states) is a state at which the system does not change its behaviour in time. (no evolution in time → derivative = 0)

$$\begin{cases} f_1(x_1, \dots, x_N) = 0 \\ \vdots \\ f_N(x_1, \dots, x_N) = 0 \end{cases}$$

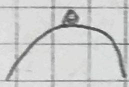
stable

small disturbances — // — damp out in time



unstable

small disturbances away from the steady state grow with time



Linear stability analysis

$N = 2$

$$(1)\quad \begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Rightarrow (x_0, y_0)$$

$\tilde{x}, \tilde{y}$ → small disturbances

$$(2)\quad \begin{aligned} x &= x_0 + \tilde{x} \\ y &= y_0 + \tilde{y} \end{aligned}$$

We substitute (2) into (1) and apply Taylor's expansion to g and f in the vicinity of (x_0, y_0).

We neglect square terms and all other terms with order larger than 2.

$x - x_0$

$$\frac{d\tilde{x}}{dt} = \underbrace{f(x_0,y_0)}_{=0} + \underbrace{\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}}_{A} \cdot \tilde{x} + \underbrace{\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}}_{B} \tilde{y}$$

$$\frac{d\tilde{y}}{dt} = \underbrace{g(x_0,y_0)}_{=0} + \underbrace{\frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}}_{C} \tilde{x} + \underbrace{\frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}}_{D} \tilde{y}$$

This are the equations for the evolution of the small disturbances in the vicinity of the fixed point

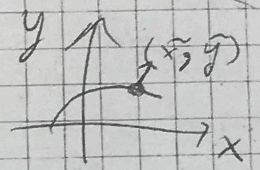
Taylor series
Taylor's expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k =$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x} + B\tilde{y}$$

$$\frac{d\tilde{y}}{dt} = C\tilde{x} + D\tilde{y}$$

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$



$$\tilde{x} = \gamma_1 e^{\lambda t}$$
$$\tilde{y} = \gamma_2 e^{\lambda t}$$

We substitute the solution into the linear equations

$$\begin{cases}(A-\lambda)\gamma_1 + B\gamma_2 = 0\\ C\gamma_1 + (D-\lambda)\gamma_2 = 0\end{cases}$$

In order to have solutions (the condition) the determinant should be zero!

Jacobian matrix

$$\begin{vmatrix}A-\lambda & B\\ C & D-\lambda\end{vmatrix} = 0$$

characteristic equation

$$\lambda^2 - (A+D)\lambda + AD - BC = 0$$

$$\underbrace{A+D}_{\text{trace of a matrix}} = S \quad (\text{"Spur"}) \qquad \underbrace{AD-BC}_{\text{determinant of the matrix}} = J$$

$$\boxed{\lambda^2 - S\lambda + J = 0} \qquad \lambda_{1,2} = \frac{S}{2} \pm \sqrt{\frac{S^2}{4} - J}$$

$N=2 \Rightarrow$ phase space dimention is $2 \Rightarrow 2$ eigenvalues

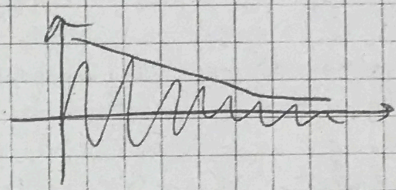
$$\tilde{x} \sim e^{\lambda_1 t}, \quad \tilde{y} \sim e^{\lambda_2 t}$$

$\lambda_{1,2}$ - real and negative $\Rightarrow$ stable fixed point

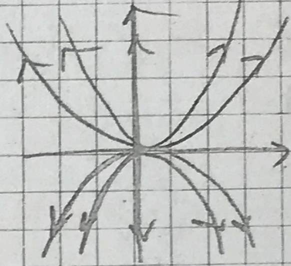
$\lambda_{1,2}$ - complex $\rightarrow$ $\lambda = \alpha + i\omega$

and negative real parts $\Rightarrow$ stable fixed point (focus)

$$\tilde{x} \sim e^{\alpha t} \underbrace{e^{j\omega t}}_{\text{rotation}}$$

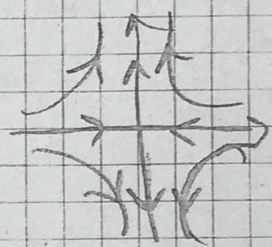


1) λ_1, λ_2 - real and positive
$\lambda_1 > 0$
$\lambda_2 > 0$



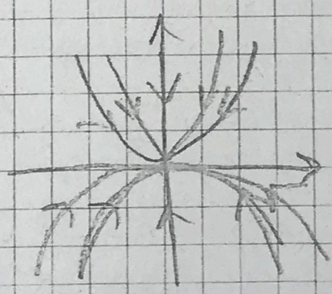
unstable node

2) λ_1, λ_2 - real
$\lambda_1 > 0$
$\lambda_2 < 0$



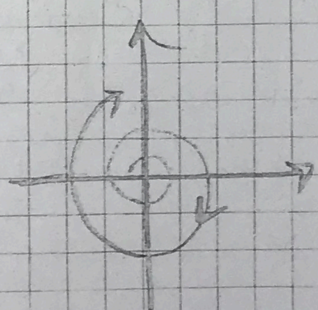
saddle

3) λ_1, λ_2 real
$\lambda_1 < 0$
$\lambda_2 < 0$



stable node

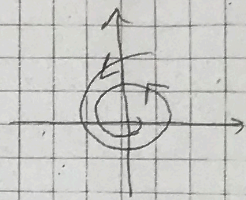
4) λ_1, λ_2 - complex
$\mathrm{Re}\ \lambda_1, \lambda_2 > 0$



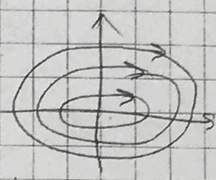
unstable focus

5) λ_1, λ_2 - complex

$Re\ \lambda_1 \lambda_2 < 0$ — stable focus



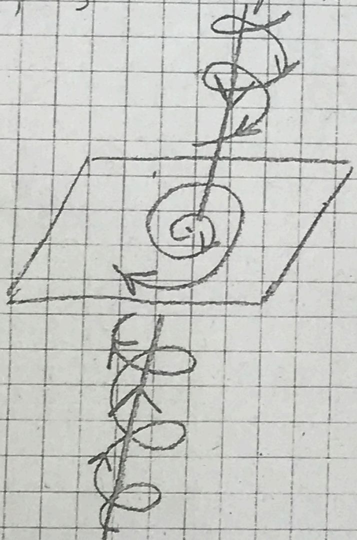
6) λ_1, λ_2 - imaginary — center



$N = 3$ saddle - focus

λ_1 - real and $\lambda_1 < 0$

λ_2, λ_3 - complex, $Re\ \lambda_{2,3} > 0$



λ_1 - real and $\lambda_1 > 0$

$\lambda_{2,3}$ complex

$Re\ \lambda_{2,3} < 0$