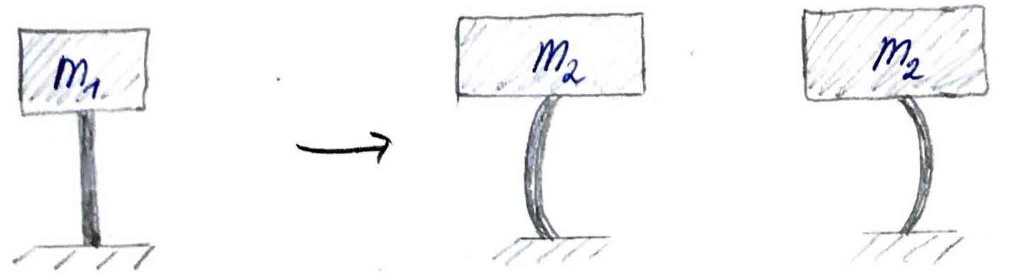
Lecture 4
30.04.2019

3) Pitchfork bifurcation

is common in physical problems that have a symmetry.

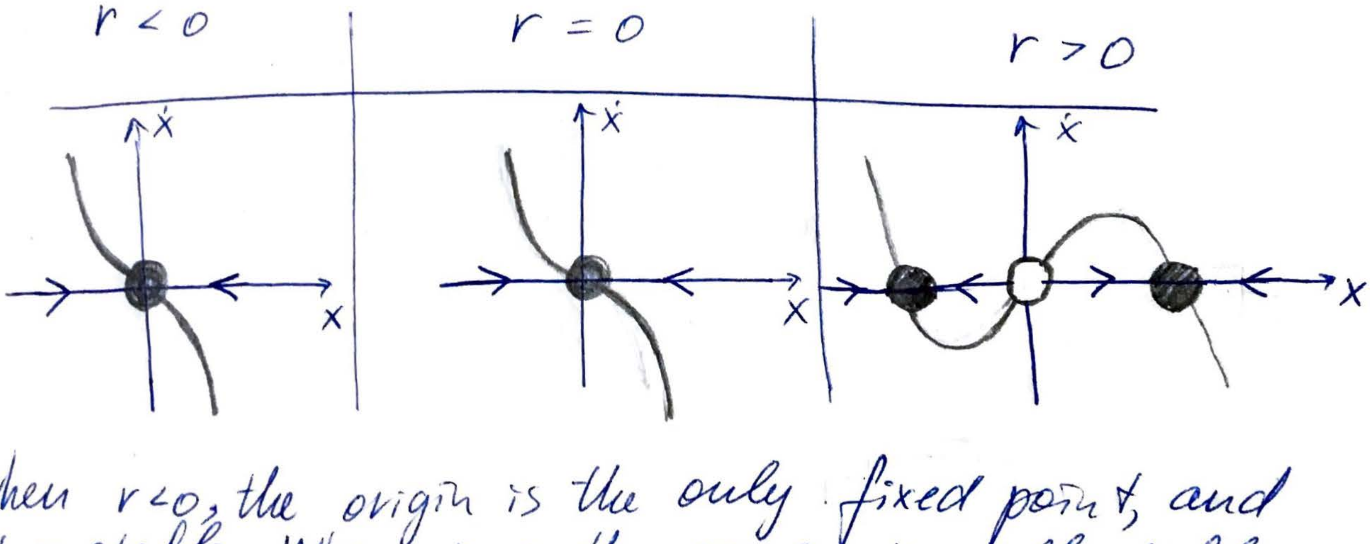
Example : beam buckling



Supercritical pitchfork bifurcation

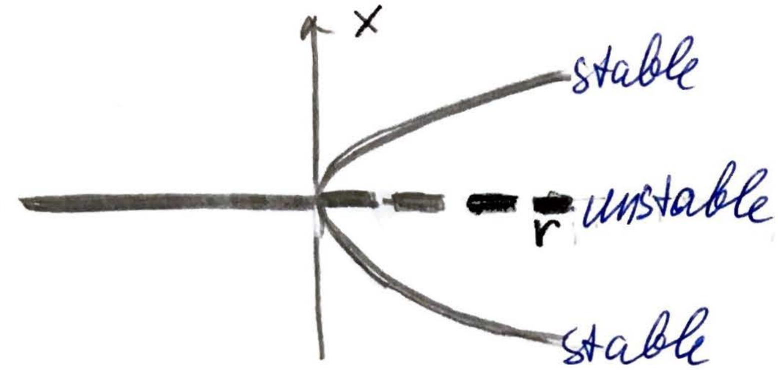
normal form $\boxed{\dot{x} = rx - x^3}$

This equation is invariant under the change of variables $x \to -x$. This invariance is the mathem. expression of the left-right symmetry.



When $r<0$, the origin is the only fixed point, and it is stable. When $r=0$ the origin is still stable, but much more weakly. When $r>0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.

Bifurcation diagram



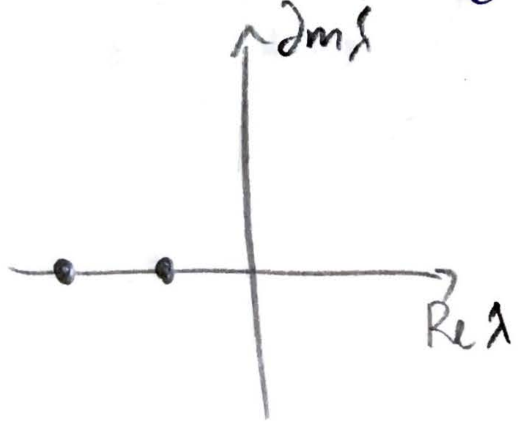
More generally, saddle-node, transcritical, and pitchfork bifurcations are all examples of <u>zero-eigenvalue bifurcations</u> (they occur when one of the eigenvalues equals zero). Such bifurcations always involve the collision of two or more fixed points.

Hopf bifurcations

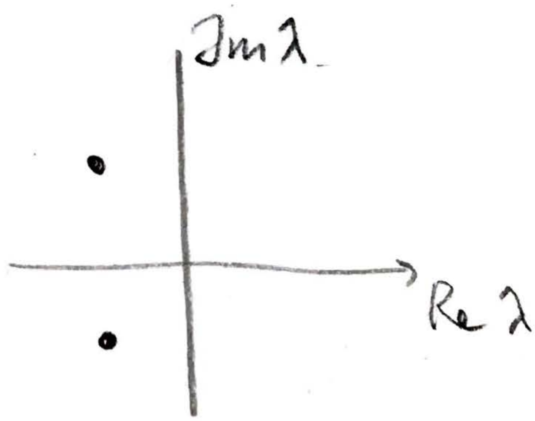
Suppose a 2D system has a stable fixed point. What are all the possible ways it could lose stability as parameter μ varies?

$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$ One has to consider the eigenvalues

Since the fixed point is stable, λ_1, λ_2 must both lie in the left half-plane $\mathrm{Re}\,\lambda < 0$.



λ_1, λ_2 are both real and negative



λ_1, λ_2 are complex conjugates

To destabilize the fixed point, we need one or both of the eigenvalues to cross into the right half-plane as μ varies.

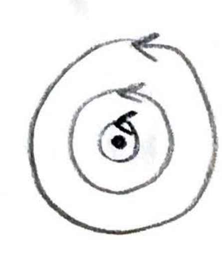
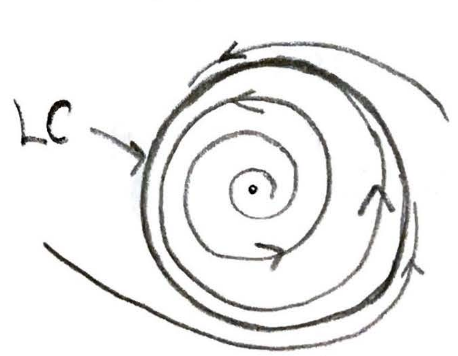
Supercritical Hopf bifurcation (HB)

In terms of flow in phase space, a supercritical HB occurs when a stable spiral changes into an unstable spiral surrounded by a small limit cycle (LC). HB occurs in phase spaces of any dimension $n \geq 2$.

Example
$$\boxed{\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2 \end{aligned}}$$

μ - controls the stability of the fixed point at the origin;
ω - frequency of osc-ions;
b - the dependence of frequency on amplitude for large amplitude oscillations.

Phase space

$\mu < 0$ (before the HB)	$\mu = 0$	$\mu > 0$ (after HB)
$\text{Re}\,\lambda_{1,2} < 0$	$\lambda_{1,2} = \pm i\omega$	$\text{Re}\,\lambda_{1,2} > 0$
		
stable focus	center	unstable focus & stable limit cycle

To see how the eigenvalues behave during the bifurcation we rewrite the system:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\Rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = (\mu r - r^3)\cos\theta -$$

$$- r(\omega + b r^2)\sin\theta = (\mu - [x^2+y^2])x -$$

$$-(\omega + b[x^2+y^2])y = \mu x - \omega y + \text{cubic terms}$$

and similarly

$$\dot{y} = \omega x + \mu y + \text{cubic terms}$$

The Jacobian at the origin is

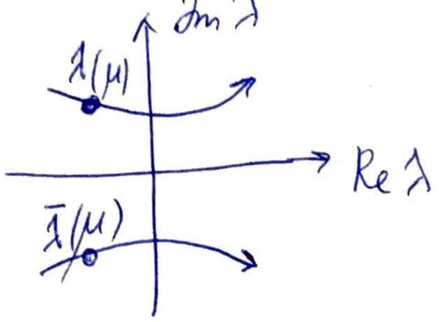
$$A = \begin{pmatrix} x & -\omega \\ \omega & \mu \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2} = \mu \pm i\omega$. Therefore, the eigenvalues cross the imaginary axis from left to right as μ increases from negative to positive values.

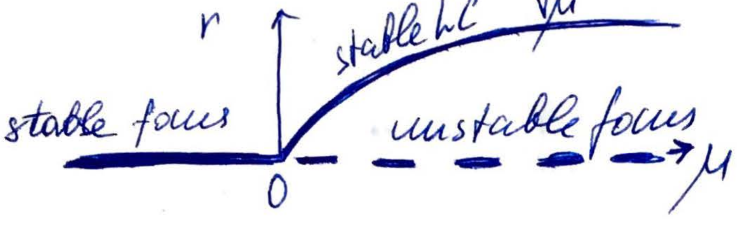
<u>Assumptions</u>/idealized example

⊛ in HB in practice, the LC can be elliptical, not circular, and, therefore, our example is only typical topologically, not geometrically.

⊛ In our idealized case the eigenvalues move on horizontal lines as μ varies, i.e., $\operatorname{Im}\lambda$ is independent of μ. Normally the eigenvalues would follow a curvy path and cross the imaginary axis with nonzero slope.

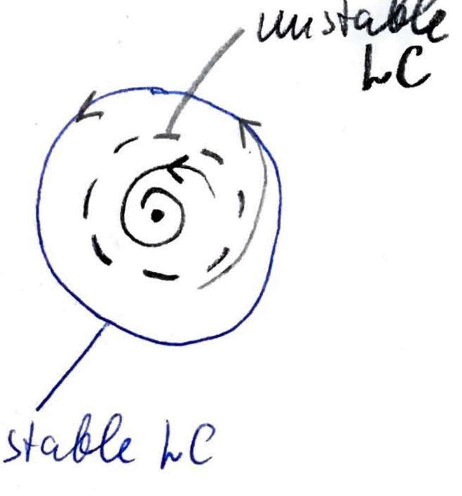
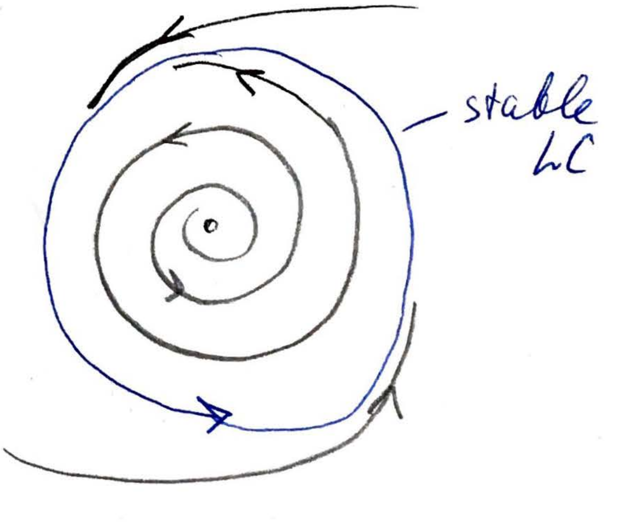


Bifurcation diagram (supercritical HB)



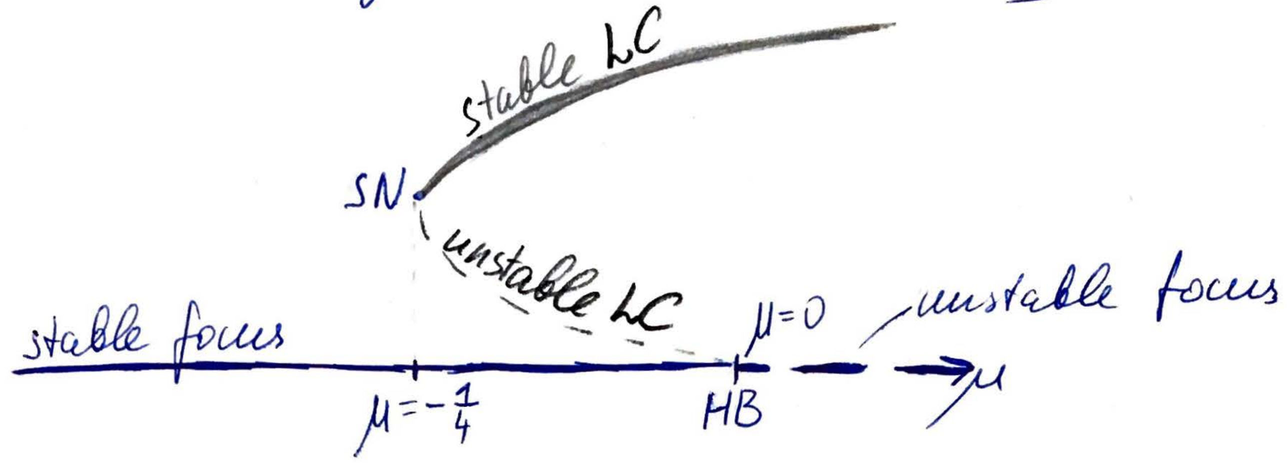
Subcritical HB

$$\boxed{\begin{aligned} \dot{r} &= \mu + r^3 - r^5 \\ \dot{\theta} &= \omega + b r^2 \end{aligned}}$$

$\mu < 0$ (before HB)	$\mu > 0$ (after HB)
	
stable focus in the origin	unstable focus in the origin

Note that the system exhibits hysteresis: once large-amplitude oscillations have begun, they cannot be turned off by bringing μ back to zero. In fact, the large oscillations will persist until $\mu = -\frac{1}{4}$ where the stable and unstable cycles collide and annihilate.

bifurcation diagram (subcritical HB).



HB - Hopf bifurcation

SN - saddle-node bifurcation of limit cycles

<u>Applications</u> the subcritical case is always much more dramatic, and potentially dangerous in engineering applications

* aeroelastic flutter and other vibrations of airplane wings (Dowell & Ilgamova 1988, Thompson & Stewart 1986)

* instabilities of fluid flows (Drazin & Reid 1981)

* dynamics of nerve cells (Rinzel & Ermentrout 1989)