3. Synchronization of periodic oscillators by periodic external action

14 May 19
Lecture 6

Synchronization

↙ external ↘ mutual

3.1 Phase dynamics

We consider the action of a weak periodic force on periodic self-sustained oscillators

The main idea: small force influences only the phase, not the amplitude => we can describe the dynamics with a phase equation.

In its derivation we follow the method developed by Malkin [1956] and Kuramoto [1984].
Although the method is quite general, the resulting phase equation is very simple and easy to investigate.

=> we can describe many important properties of sync analytically.

3.1.1 A limit cycle and the phase of oscillations

Geometrical image of periodic self-sustained oscillator is a limit cycle.

Suppose an oscillator generates a periodic process $x(t)$.

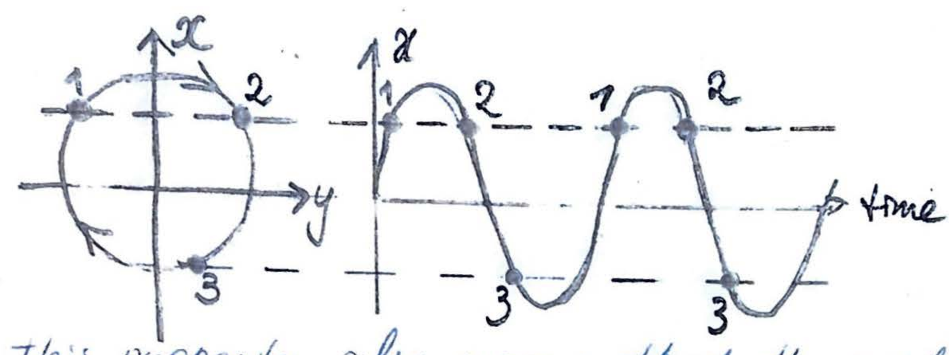
$x(t)$ can refer to
* the angle of the clock's pendulum
* intensity of light in a laser
* electrical current through a resistor in a vacuum tube generator

We want to describe the state of the oscillator at a certain time. It is not enough to know only x. Quite often two variables are enough: x and y.

For a pendulum clock: angle + angular velocity.

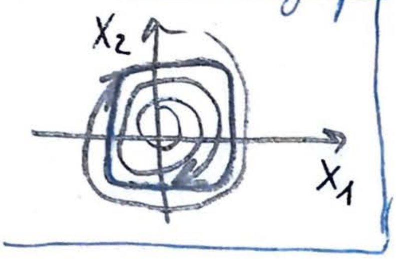
These variables are called coordinates in the phase space (state space) and the $y(t)$ vs. $x(t)$ plot is called the phase portrait of the system. The point with coordinates (x, y) is often called a phase point.

The oscillation is periodic => it repeats itself after the period T => $x(t)$ corresponds to a closed curve in the phase plane, called the limit cycle.



After perturbation the original rhythm is restored $\Rightarrow$ the phase point returns to a limit cycle

This property also means that the oscillations do not depend on I.C. (at least in some range), or on how the system was initially put into the motion.



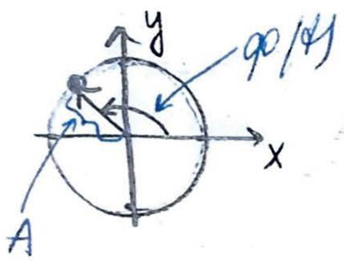
$x(t) = A \sin(\omega_0 t + \varphi_0)$

$\omega_0 \to$ the angular frequency, $\omega_0 = \frac{2\pi}{T}$

$A \to$ amplitude

$\varphi(t) = \omega_0 t + \varphi_0 \to$ phase

$\varphi_0 \to$ initial phase



Consider a general M-dimensional ($M \geq 2$) system of ODE

$$\frac{d\bar{x}}{dt} = \bar{f}(\bar{x}), \quad \bar{x} = (x_1, \dots, x_M), \quad (3.1)$$

and suppose that this system has a stable periodic (T_0) solution $\bar{x}_0(t) = \bar{x}_0(t_0 + T_0)$. In the phase space this corresponds to a limit cycle – isolated closed attractive trajectory.

The point in the phase space moving along the cycle represents the self-sustained oscillations.

The most popular classical example of a self-oscillating system is the van der Pol equation.

$$\ddot{x} - 2\mu\dot{x}(1 - \beta x^2) + \omega_0^2 x = 0 \quad (3.2)$$

For small μ it performs sin-oscillations and for large μ it is of relaxation type.

Our aim is to describe the motion in terms of the phase:

the phase φ grows monotonically in the direction of the motion and gains 2π during each rotation; moreover, we assume that the phase grows uniformly in time:

$$\frac{d\varphi}{dt} = \omega_0 \quad (3.3), \qquad \omega_0 = \frac{2\pi}{T_0} \quad (3.4),$$

where ω_0 is the natural frequency

3.1.2. Small perturbations and isochrones

We now consider the effect of a small external periodic force on the self-sustained oscillations:

$$\frac{d\bar{x}}{dt} = \bar{f}(\bar{x}) + \varepsilon \bar{p}(\bar{x},t), \quad (3.5)$$

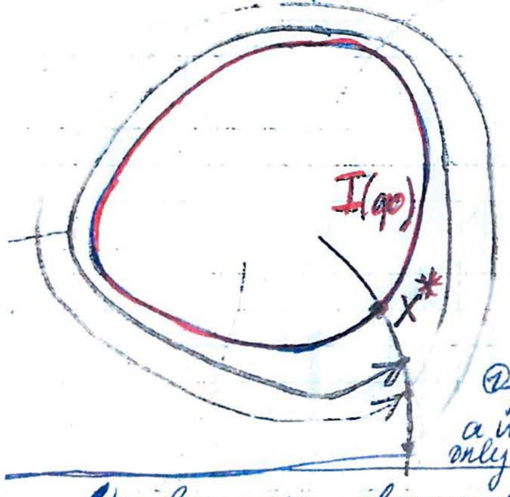
where the force has a period T, which is in general different from T_0:

$$\varepsilon \bar{p}(\bar{x},t) = \varepsilon \bar{p}(\bar{x}, t+T); \quad \varepsilon \text{ is a small parameter.}$$

The external force is small
The cycle is stable $\Rightarrow$ the trajectory only slightly deviates from the original one $\bar{x}_0(t)$

Perturbations in the directions transverse to the cycle are small. Contrary to this, the phase perturbation can be large: the force can easily drive the phase point along the cycle.

Let us define isochrones $I(\varphi_0)$ in the vicinity of the stable limit cycle.[1]

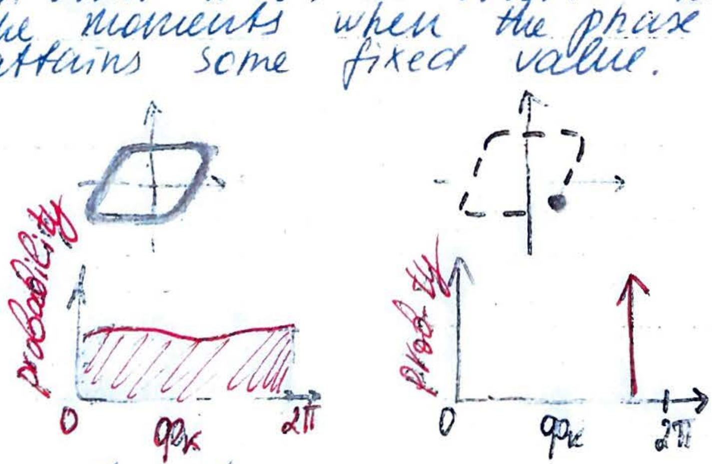


Let us observe the dynamical system (3.1) stroboscopically, with the time interval being exactly the period of the limit cycle T_0. Thus, we get from (3.1) a mapping:

$$\boxed{\bar{x}(t) \rightarrow \bar{x}(t+T_0) \equiv \Phi(\bar{x})} \rightarrow$$

[1] The main idea is to define the phase variable in such a way that it rotates uniformly according to (3.1) not only on the cycle, but in its neighborhood as well.

Stroboscopic observation $\rightarrow$ we observe the phase not continuously, but at the times $t_k = k \cdot T$, where T is the period of external forcing, and $k = 1, 2, \ldots$
In other words we observe the driven system at the moments when the phase of the external force attains some fixed value.



If the oscillator is synchronized by an external periodic force, then a point is always found in the same position on the cycle. If the frequency of the oscillator differs from the freq of forcing the point can be found at any position.

φ_k is the phase

It is easy shift defi...

→ the mapping has all points on the limit cycle as fixed points, and all points in the vicinity of the cycle are attracted to it. Let us choose a point $\bar{x}^*$ on the cycle and consider all the points in the vicinity that are attracted to $\bar{x}^*$ under the action of Φ.

They form a hypersurface I which is $(M-1)$-dimensional. It is called an isochrone → crossing the limit cycle at x^*.

We can draw an isochrone hypersurface at every point on the cycle. We now extend the definition of the phase to the vicinity of the limit cycle, demanding that the phase is constant on each isochrone $I(\varphi_0)$.

3.1.3. An example: complex amplitude equation

We now consider a particular example of a system with a limit cycle and define its phase and isochrones.

Stuart-Landau oscillator:

$$\frac{dA}{dt} = (1+i\eta)A - (1+i\alpha)|A|^2 A \qquad (3.6)$$

This is a first-order dif. equation for the complex variable A.

We can rewrite it in polar coordinates, $A = Re^{i\theta}$:

$$\frac{dR}{dt} = R(1-R^2) \qquad (3.7)$$

$$\frac{d\theta}{dt} = \eta - \alpha R^2 \qquad (3.8)$$

One can solve it: the limit cycle here is a unit circle $R=1$ and the solution with arbitrary I.C. $R_0 = R(0)$, $\theta_0 = \theta(0)$ reads:

$$R(t) = \left[1 + \frac{1-R_0^2}{R_0^2} e^{-2t}\right]^{-1/2}, \qquad (3.9)$$

$$\theta(t) = \theta_0 + (\eta - \alpha)t - \frac{\alpha}{2} \ln\left(R_0^2 + (1-R_0^2)e^{-2t}\right).$$

On the limit cycle the angle variable θ rotates with a constant velocity $\omega_0 = \eta - \alpha$ and, hence, can be taken as the phase φ_0. However, if the initial amplitude deviates from unity, an additional phase shift occurs (due to the term proportional to α in (3.8)).

It is easy to see from (3.9) that the additional phase shift is equal to $-\alpha \ln R_0$. Thus, the phase can be defined on the whole plane (R, θ) as

$$\varphi(R, \theta) = \theta - \alpha \ln R \qquad (3.10)$$

One can check that this phase indeed rotates uniformly.

$$\frac{d\varphi}{dt} = \frac{d\theta}{dt} - \alpha \frac{\dot{R}}{R} = \eta - \alpha$$

The isochrones are the lines of constant phase φ; in the (R, θ) plane these are logarithmic spirals:

$$\theta - \alpha \ln R = \mathrm{const}$$

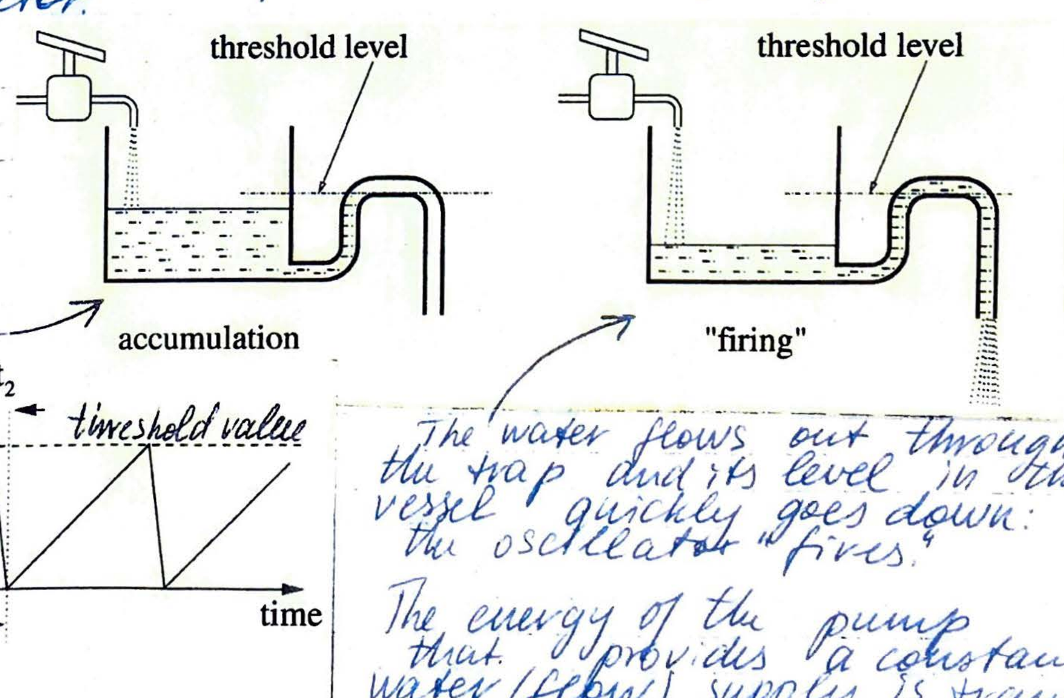
For $\alpha = 0$ we have lines $\theta = \mathrm{const} = \varphi$ (instead of spirals).

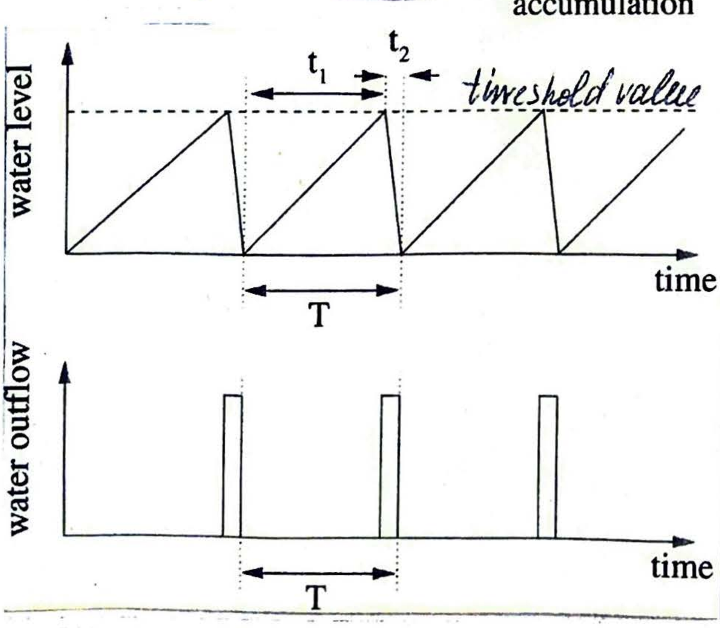
Relaxation oscillators

The essential feature of these systems is the presence of two time scales: within each cycle there are intervals of slow and fast motion. The form of the oscillations is therefore very far from the simple sine wave; it rather resembles a sequence of pulses.

Mechanical model of a relaxation (integrate-and-fire) oscillator.

The water slowly fills the vessel until the threshold value is reached → this part of the cycle is "integration"





The water flows out through the trap and its level in the vessel quickly goes down: the oscillator "fires".

The energy of the pump that provides a constant water (flow) supply is transformed into the oscillation of the water level in the vessel.

Although this is a "toy" model it captures the main features of real systems.

As within each cycle water is slowly accumulated and then almost instantly removed, such systems are called integrate-and-fire or accumulate-and-fire oscillators. If we plot the level of water and its flow vs. time, the intervals of the slow and fast motions can be clearly seen: t_1 - water level slowly grows; t_2 - water level is quickly reset to 0.