3.1.4. The equation for the phase dynamics

We have defined the phase in some neighborhood of the L.C. $\Rightarrow$ we can write (3.3) in this vicinity as:

$$\frac{d\varphi(\bar{x})}{dt} = \omega_0 \quad (3.11)$$

As the phase is a smooth function of the coordinates $\bar{x}$, we can represent its time derivative as

$$\frac{d\varphi(\bar{x})}{dt} = \sum_k \frac{\partial\varphi}{\partial x_k}\frac{dx_k}{dt}, \quad (3.12)$$

which gives, together with (3.1), the relation

$$\sum_k \frac{\partial\varphi}{\partial x_k} f_k(\bar{x}) = \omega_0$$

We consider now the perturbed system (3.5). Using the "unperturbed" definition of the phase and substituting Eq. (3.5) in (3.12) we obtain

$$\frac{d\varphi(\bar{x})}{dt} = \sum_k \frac{\partial\varphi}{\partial x_k}\left(f_k(\bar{x}) + \varepsilon p_k(\bar{x},t)\right) = \omega_0 + \underbrace{+\,\varepsilon\sum_k \frac{\partial\varphi}{\partial x_k} p_k(\bar{x},t)} \quad (3.13)$$

The second term on the r.h.s. is small (proportional to ε), and the deviations of $\bar{x}$ from the L.C $\bar{x}_0$ are small too. Thus, in the first approximation we can neglect these deviations and calculate the r.h.s. on the L.C.:

$$\frac{d\varphi(\bar{x})}{dt} = \omega_0 + \varepsilon\sum_k \frac{\partial\varphi(\bar{x}_0)}{\partial x_k} p_k(\bar{x}_0,t) \quad (3.14)$$

Because the points on the L.C. are in one-to-one correspondence with the phase φ, we get a closed equation for the phase:

$$\boxed{\frac{d\varphi}{dt} = \omega_0 + \varepsilon Q(\varphi,t) \quad (3.15)}$$

with $Q(\varphi,t) = \sum \dfrac{\partial\varphi(\bar{x}_0)}{\partial x_k} p_k(\bar{x}_0(\varphi),t)$.

Q is a 2π-periodic function of φ and T-periodic function of t.

3.1.5. An example: forced complex amplitude equations

As an example, we periodically force the nonlinear oscillator described by Eq. (3.6), which we rewrite as a system of real equations:

$$\frac{dx}{dt} = x - \eta y - (x^2+y^2)(x - \alpha y) + \varepsilon \cos \omega t,$$

$$\frac{dy}{dt} = y + \eta x - (x^2+y^2)(y + \alpha x).$$

$\boxed{\varphi(R,\theta) = \theta - \alpha \ln R}$

Rewriting the expression for the phase (3.10) as

$$\varphi = \tan^{-1}\frac{y}{x} - \frac{\alpha}{2}\ln(x^2+y^2),$$

we obtain a particular case of the phase equation (3.15):

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon \frac{\partial \varphi}{\partial x}\cos\omega t = \eta - \alpha - \varepsilon(\alpha\cos\varphi + \sin\varphi)\cos\omega t,$$

yielding, with $\tan\varphi_0 = 1/\alpha$,

$$\frac{d\varphi}{dt} = \eta - \alpha - \varepsilon\sqrt{1+\alpha^2}\cos(\varphi - \varphi_0)\cos\omega t \qquad (3.16)$$

Equation (3.15) is the basic equation describing the dynamics of the phase of a self-sustained periodic oscillator in the presence of a small periodic external force.

Self-sustained oscillators: main features

Dissipation, stability, nonlinearity.

$\boxed{\text{Dissipation}}$ Natural systems dissipate their energy

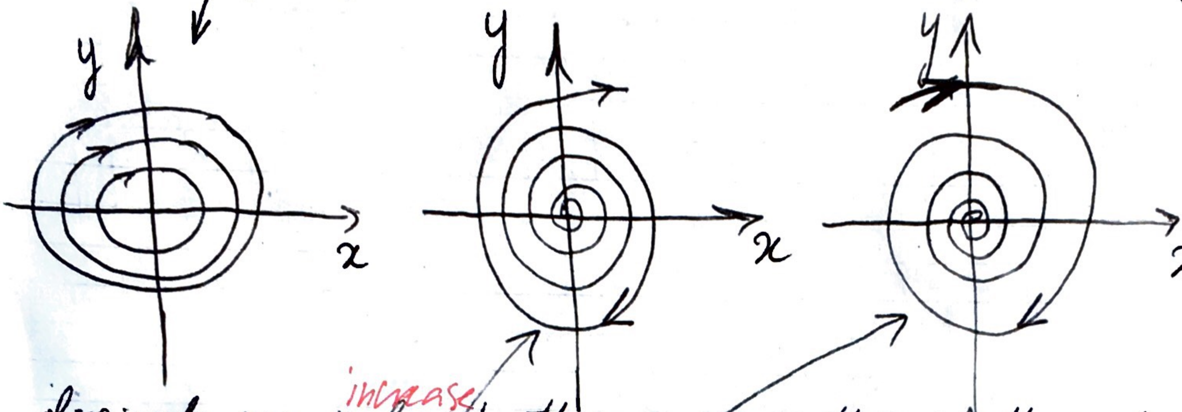
Without a constant supply of energy into the system, the oscillations would decay.

Therefore, self-sustained oscillators must possess an internal energy source.

* potential energy of the lifted weight or compressed spring (in the pendulum clock)
* electrical power supply (modern clock)
* chemical reactions (heart, firefly)

Stability with respect to perturbations

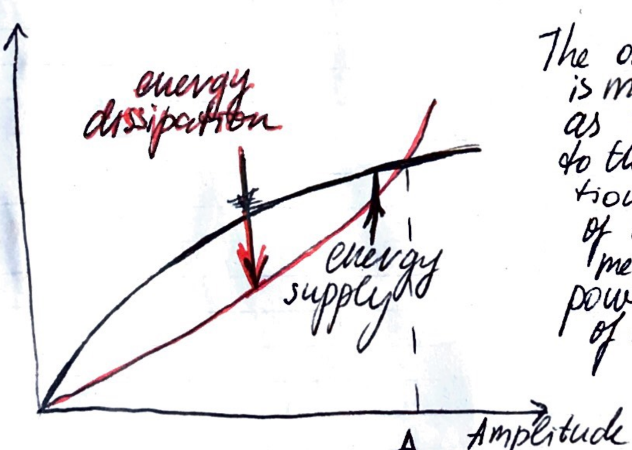
The property of stability of oscillations with respect to perturbations also distinguishes self-sustained oscillators from conservative ones. Periodic motions in conservative systems can be described in the phase plane by a family of closed curves. These systems neigther



dissipate nor increase replenish their energy then if the system is perturbed, it remains perturbed. The altered value of the oscillation amplitude is preserved as well $\Rightarrow$ conservative periodic systems do not "forget" the I.C.

Nonlinearity is also essential for the maintenance of stable limit cycle oscillations. Stability means that a periodic motion exists only with a certain amplitude. This can not be obtained within the class of models that are described by $\boxed{\text{linear}}$ equations: if a solution $x(t)$ of a linear system is periodic, then for any factor a $a \cdot x(t)$ is a periodic solution as well $\Rightarrow$ the linear system exhibiting periodic oscillations is necessarily a conservative one

Linear systems with dissipation (or a source of energy) admit infinitely growing or decaying solutions, but never a limit cycle.



The onset of stable oscillations is mathematically described as the attraction of a trajectory to the L.C. Physically this attraction can be understood in terms of energy. Whatever the exact mechanism of dissipation and power supply is the energy of some source, usually nonoscillating, is transformed into the oscillatory motion

Typically, the larger the amplitude of oscillations, the more energy is taken from the source. From the other side the amount of dissipated energy also depends on the amplitude.

[This consideration is valid for systems having the phase portrait  where trajectory spirals towards the L.C., so that we can speak of oscillation with slowly varying amplitude.]

The interplay of energy dissipation and energy supply determines the amplitude of the oscillations A.

If there is no oscillation (zero amplitude), there is no dissipation and no energy is taken from the source - therefore both these functions pass through the origin. The intersection of these curves determines the amplitude A of stationary oscillations, in this point the energy supplied to the system exactly compensates the dissipated energy. We note that these two curves can intersect in this way and simultaneously pass through the origin only if the system is nonlinear, i.e. described by nonlinear diff. equations.

What are the main features of self-sustained oscillators?

What are the examples of internal energy sources?

Why linear dissipative systems can not have a limit cycle?

How can the attraction of a trajectory to a L.C. be explained in terms of energy?

Can a linear system exhibit periodic oscillations?