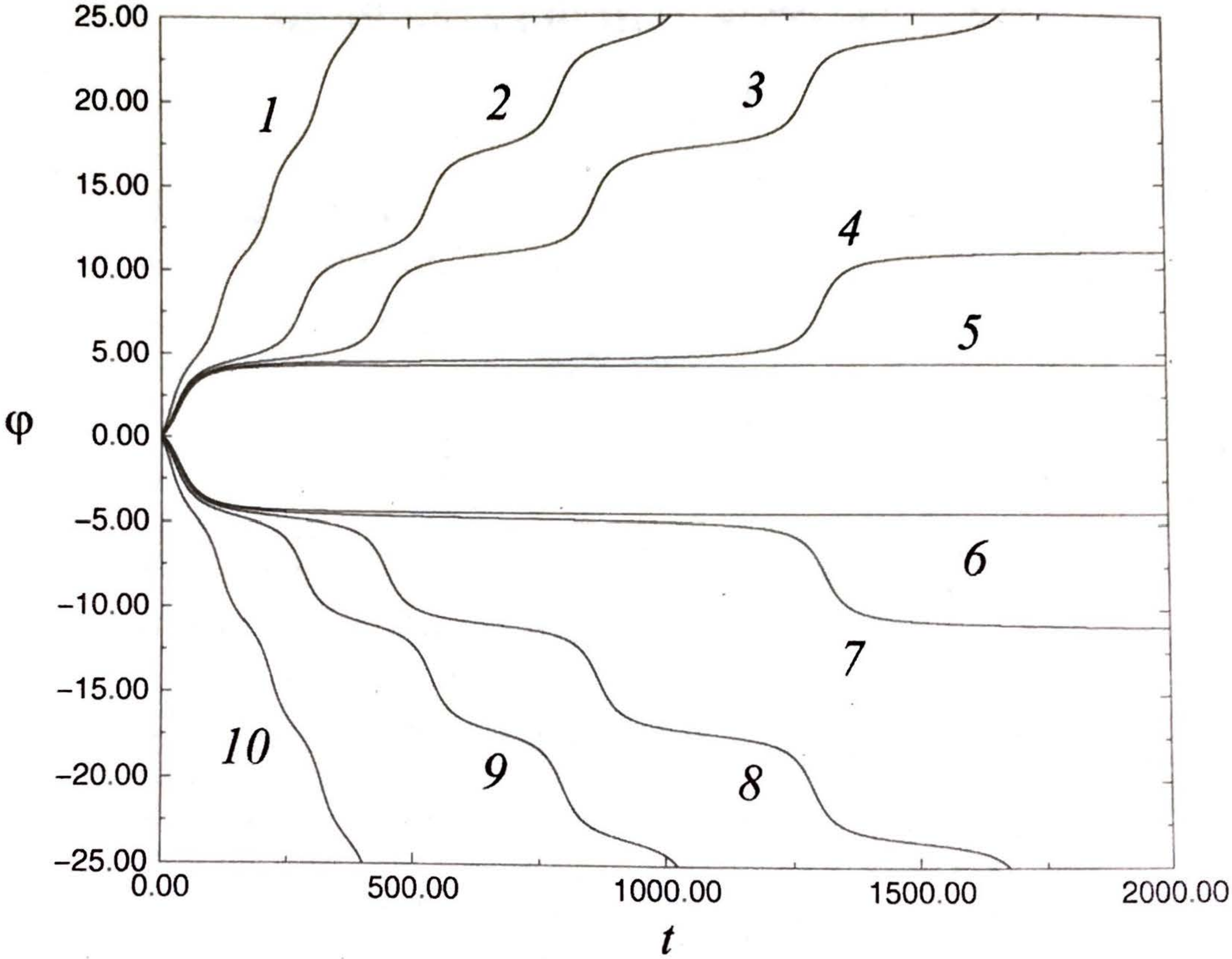
Lecture 9 04.06.2019

Fig. 4.1 : $\varepsilon = 0.1$, $\beta = 0.01$



Temporal phase dynamics $\varphi(t)$ for $\varepsilon = 0.1$ and $\beta = 0.01$ is shown in Fig 4.1. for different values of the frequency detuning:

$\Delta = \mp 0.07$ (lines 1 and 10)

$\Delta = \mp 0.04$ (lines 2 and 9)

$\Delta = \mp 0.035$ (lines 3 and 8)

$\Delta = \mp 0.032$ (lines 4 and 7)

$\Delta = \mp 0.03$ (lines 5 and 6)

For small detuning, the phase is constant (lines 5 and 6 for $\Delta = \mp 0.03$). As the frequency detuning grows and when $|\Delta|$ exceeds a certain critical value $|\Delta_c|$, the phase behavior $\varphi(t)$ changes qualitatively. In fact, its value starts varying in time.

According to the sign of $\Delta$ (if the freq. of the forcing is greater or less than the natural frequency), the phase $\varphi$ either decreases or increases in time.

For a small subcriticality $|\Delta - \Delta_c|$, the the time series $\varphi(t)$ shows long intervals during which the phase is nearly constant. These long intervals intermingle with relatively short time intervals where the phase changes by $2\pi$. As the supercriticality grows, the intervals of constant phase decrease, and the mean rate of phase change increases.

Thus, there is an interval of values of the detuning $|\Delta| < |\Delta_c|$ where the phase is constant, $\varphi(t) = \text{const}$, and its derivative (the rate of phase change) is zero. This means that the oscillator oscillates periodically at the frequency of the external force $\Rightarrow$ sync!

Outside the sync region the phase changes in time and the oscillations become quasiperiodic. The mean rate of the phase change $\langle \dot{\varphi}(t) \rangle$ defines the second frequency, the so-called beat frequency.

In the phase approximation, sync motions correspond to fixed points of the dynamical system (4.9):

$$\frac{d\varphi}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi \qquad (4.9)$$

Regimes of sync are related to stable steady states.

$$-\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi = 0 \quad (4.10)$$

We find fixed points:

$$\varphi_1 = \arcsin \frac{2\Delta\sqrt{\varepsilon}}{\beta} \quad (4.11)$$

$$\varphi_2 = \pi - \arcsin \frac{2\Delta\sqrt{\varepsilon}}{\beta} \quad (4.12)$$

They exist if $|2\Delta\sqrt{\varepsilon}/\beta| \leq 1$ or $|\Delta| \leq \beta/2\sqrt{\varepsilon}$. If we fix $\varepsilon$ the region of existence of the fixed points in the parameter plane $(\beta, \Delta)$ is bounded by the line given by the equation: $\beta = 2\sqrt{\varepsilon}\,|\Delta|$. (4.13)

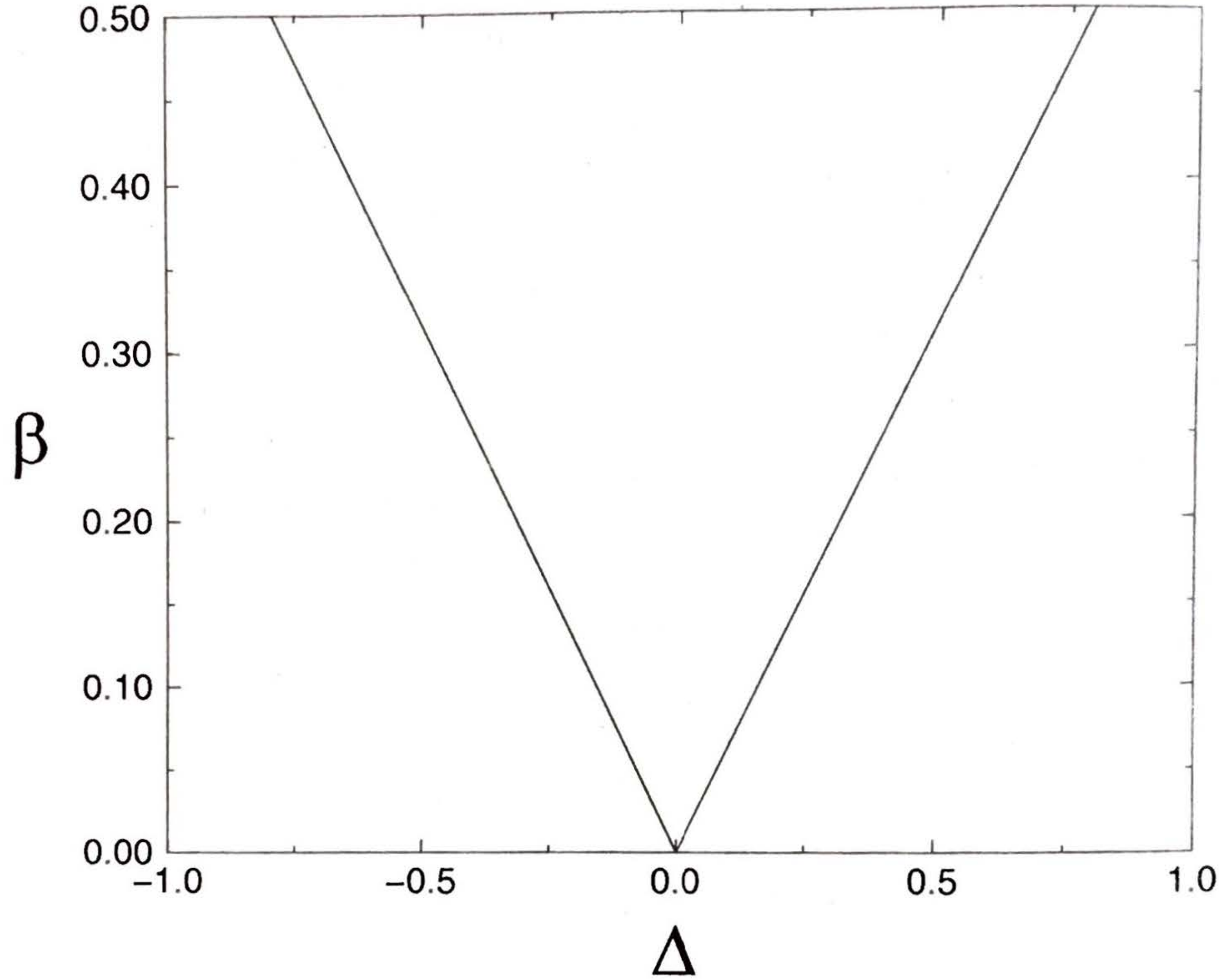

0.50
0.40
0.30
0.20
0.10
0.00
β
−1.0
−0.5
0.0
0.5
1.0
Δ

Fig. 4.2 Sync region in the plane of parameters $\Delta$ and $\beta$ for $\varepsilon = 0.1$.

Let us analyze the stability of the fixed points. The behavior of the system (4.9) is considered in the neighbourhood of fixed points $\varphi_i$ $(i=1,2)$ in a linear approx. We represent the dynamical variable $\varphi(t)$ in the form: $\varphi(t) = \varphi_i + \tilde{\varphi}(t)$, where $\tilde{\varphi}(t)$ is a small deviation from a fixed point $\varphi_i$.

We rewrite (4.9) as follows:

$$\frac{d}{dt}(\varphi_i + \tilde{\varphi}) = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin(\varphi_i + \tilde{\varphi}),$$

$$\frac{d(\varphi_i)}{dt} = 0 \quad \text{( in the fixed point)}$$

$$\frac{d\tilde{\varphi}}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}}(\sin\varphi_i \cos\tilde{\varphi} + \sin\tilde{\varphi}\cos\varphi_i)$$

Expanding $\cos\tilde{\varphi}$ and $\sin\tilde{\varphi}$ in Taylor series and taking the first-order terms in $\tilde{\varphi}$, we obtain the linearized equation:

$$\frac{d\tilde{\varphi}}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}}\left(\sin\varphi_i + \tilde{\varphi}\cos\varphi_i\right)$$

From the definition of the fixed point:

$$-\Delta + \frac{\beta}{2\sqrt{\varepsilon}}\sin\varphi_i = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \frac{d\tilde{\varphi}}{dt} = \left(\frac{\beta}{2\sqrt{\varepsilon}}\cos\varphi_i\right)\tilde{\varphi} \qquad (4.14)$$

solution $\tilde{\varphi}(t) \sim \exp\left[\left(\frac{\beta}{2\sqrt{\varepsilon}}\cos\varphi_i\right)t\right]$ (4.15)

The stability of fixed points depends on the sign of $\cos\varphi_i$. If $\cos\varphi_i < 0$, then the small deviation decays in time and the fixed point is stable. If $\cos\varphi_i > 0$ $\Rightarrow$ the fixed point is unstable.