3.6 Eichinvarianz

Darstellung der Felder $\underline{E}$, $\underline{B}$ durch Potenziale $\phi(\underline{r},t)$, $\underline{A}(\underline{r},t)$:

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A} \quad , \quad \underline{B} = \underline{\nabla}\times\underline{A}}$$

<u>Frage</u>: allgemeinste Trafo $\phi \to \phi'$, $\underline{A} \to \underline{A}'$, welche $\underline{E}$, $\underline{B}$ invariant läßt?

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A} \overset{!}{=} -\underline{\nabla}\phi' - \frac{\partial}{\partial t}\underline{A}'$$

$$\underline{B} = \underline{\nabla}\times\underline{A} \overset{!}{=} \underline{\nabla}\times\underline{A}' \Rightarrow \underline{A}' = \underline{A} + \underline{\nabla}G(\underline{r},t)$$

(da $\underline{\nabla}\times\underline{\nabla}G = 0$)

$$\Rightarrow -\underline{\nabla}\phi - \cancel{\frac{\partial}{\partial t}\underline{A}} \overset{!}{=} -\underline{\nabla}\phi' - \cancel{\frac{\partial}{\partial t}}(\cancel{\underline{A}} + \underline{\nabla}G)$$

$$\Rightarrow \underline{\nabla}\left(\phi' - \phi + \frac{\partial}{\partial t}G\right) = 0$$

$\Rightarrow \phi' - \phi + \frac{\partial}{\partial t}G = g(t)$ unabh. von $\underline{r}$

Mit $F(\underline{r},t) := G(\underline{r},t) - \int_{t_0}^{t} dt'\, g(t')$

ergibt sich

$$\boxed{\begin{aligned} \underline{A}'(\underline{r},t) &= \underline{A}(\underline{r},t) + \underline{\nabla}F(\underline{r},t) \\ \phi'(\underline{r},t) &= \phi(\underline{r},t) - \frac{\partial}{\partial t}F(\underline{r},t) \end{aligned}}$$

mit <u>beliebiger</u> Eichfkt. $F(\underline{r},t)$

Durch $\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$, $\underline{B} = \underline{\nabla}\times\underline{A}$

sind die <u>homog</u>. Maxwell-Gln. automatisch erfüllt:

$$\underline{\nabla}\times\underline{E} = -\underbrace{\underline{\nabla}\times\underline{\nabla}\phi}_{0} - \frac{\partial}{\partial t}\underbrace{\underline{\nabla}\times\underline{A}}_{\underline{B}} , \quad \underline{\nabla}\cdot\underline{B} = 0$$

Die Umkehrung gilt auch:

$$\underline{\nabla}\cdot\underline{B} = 0 \quad \Longrightarrow \exists\, \underline{A} : \quad \underline{B} = \underline{\nabla}\times\underline{A} \quad (\text{konstr.!})$$

eingesetzt in

$$\underline{\nabla}\times\underline{E} = -\frac{\partial}{\partial t}\underline{B} = -\underline{\nabla}\times\frac{\partial}{\partial t}\underline{A}$$

$$\rightarrow \underline{\nabla}\times\left(\underline{E} + \frac{\partial}{\partial t}\underline{A}\right) = 0$$

$$\text{d.h. } \exists\,\phi : \underline{E} + \frac{\partial}{\partial t}\underline{A} = -\underline{\nabla}\phi$$

Wähle nun eine Eichung, so dass die <u>inhomogenen</u> Maxwell-Gln. besonders einfach werden:

<u>Ziel</u>: Entkopplung der Dgln. für $\underline{A}$ und ϕ

(i) <u>Lorentz-Eichung</u>

$$\boxed{\underline{\nabla}\cdot\underline{A} + \epsilon_0\mu_0\frac{\partial}{\partial t}\phi = 0}$$

Hiermit werden die Feldgln. entkoppelt:

a) $-\underline{\nabla}\cdot\underline{E} = \underline{\nabla}\left(\underline{\nabla}\phi + \frac{\partial}{\partial t}\underline{A}\right) = -\frac{1}{\epsilon_0}\rho$

$$\Delta\phi + \frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{A} = -\frac{1}{\epsilon_0}\rho$$

Lorentz-Eichung:

$$\boxed{\Delta\phi - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\phi = -\frac{1}{\epsilon_0}\rho}$$

b) $\frac{1}{\mu_0}\underline{\nabla}\times\underline{B} - \epsilon_0\frac{\partial}{\partial t}\underline{E} = \underline{j}$

$$\underbrace{\underline{\nabla}\times(\underline{\nabla}\times\underline{A})}_{\underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \Delta\underline{A}} + \epsilon_0\mu_0\frac{\partial}{\partial t}\left(\underline{\nabla}\phi + \frac{\partial}{\partial t}\underline{A}\right) = \mu_0\underline{j}$$

$$\Delta\underline{A} - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\underline{A} - \underline{\nabla}\underbrace{\left(\underline{\nabla}\cdot\underline{A} + \epsilon_0\mu_0\frac{\partial}{\partial t}\phi\right)}_{0 \quad \text{Lorentz-Eichung}} = -\mu_0\underline{j}$$

$$\boxed{\Delta\underline{A} - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\underline{A} = -\mu_0\underline{j}}$$

Zusammenfassung mit d'Alembert-Op. $\Box := \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.994 \times 10^8 \frac{m}{s}$ Lichtgeschwindigkeit

(= Ausbreitungsgeschw. el. magn. Wellen im Vakuum)

$$\boxed{\begin{aligned}\Box\phi &= -\frac{1}{\epsilon_0}\rho \\ \Box\underline{A} &= -\mu_0\underline{j}\end{aligned}}$$

inhomogene Wellengln. (entkoppelt!)

(ii) Coulomb-Eichung (Strahlungseichung)

$$\boxed{\underline{\nabla}\cdot\underline{A} = 0}$$

Allg.: Zerlegung von $\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$

in Longitudinalfeld $\underline{E}_\ell := -\underline{\nabla}\phi$ (wirbelfrei)

und Transversalfeld $\underline{E}_t := -\frac{\partial}{\partial t}\underline{A}$ (quellenfrei)

$\Big\}$ Def.

Tatsächlich gilt: $\underline{\nabla}\times\underline{E}_\ell = -\underline{\nabla}\times\underline{\nabla}\phi = 0$

$$\underline{\nabla}\cdot\underline{E}_t = -\frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{A} = 0$$

$\sim e^{i\underline{k}\cdot\underline{r}}$ $\quad \underline{\nabla}\times\underline{E} \to \underline{k}\times\underline{E}$

$\underline{\nabla}\cdot\underline{E} \to \underline{k}\cdot\underline{E}$

Zerlegung der Stromdichte: $\underline{j} = \underline{j}_\ell + \underline{j}_t$

mit $\underline{\nabla}\times\underline{j}_\ell = 0$, $\underline{\nabla}\cdot\underline{j}_t = 0$

Mit $\underbrace{\frac{\partial}{\partial t}\varrho}_{\epsilon_0\underline{\nabla}\cdot\underline{E}_\ell} + \underline{\nabla}\cdot\underline{j}_\ell + \underbrace{\underline{\nabla}\cdot\underline{j}_t}_{0} = 0$ folgt

$$\underline{\nabla}\cdot\left(\underline{j}_\ell + \epsilon_0\frac{\partial\underline{E}_\ell}{\partial t}\right) = 0$$

außerdem nach Def. $\underline{\nabla}\times\left(\underline{j}_\ell + \epsilon_0\frac{\partial}{\partial t}\underline{E}_\ell\right) = 0$

$$\Longrightarrow \quad \underline{j}_\ell + \epsilon_0\frac{\partial}{\partial t}\underline{E}_\ell = \text{const.} = 0$$

$$\boxed{\underline{j}_\ell = \epsilon_0\underline{\nabla}\frac{\partial\phi}{\partial t}}$$

Die Feldgln. $\Delta\phi + \frac{\partial}{\partial t}\underbrace{\underline{\nabla}\cdot\underline{A}}_{0} = -\frac{1}{\epsilon_0}\rho$

$$\underbrace{\Delta\underline{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A}}_{\Box\underline{A}} - \underline{\nabla}(\underbrace{\underline{\nabla}\cdot\underline{A}}_{0}) - \mu_0\underbrace{\epsilon_0\underline{\nabla}\frac{\partial}{\partial t}\phi}_{\underline{j}_l} = -\mu_0\underline{j}$$

$$\boxed{\begin{aligned}\Delta\phi &= -\frac{1}{\epsilon_0}\rho \\ \Box\underline{A} &= -\mu_0\underline{j}_t\end{aligned}}$$

$\Rightarrow$ longit. Felder $\triangleq$ El.statik

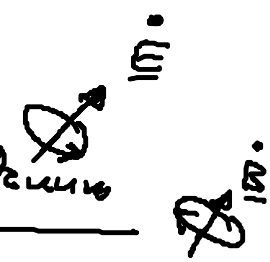
$\Rightarrow$ transv. Felder $\triangleq$ el. magn. Wellen

Coulomb-Eichung ist zweckmäßig bei Strahlungsproblemen!

Poisson-gl. für ϕ (long. Felder)
Wellen-gl. für $\underline{A}$ (transv. Felder)

4. Elektromagn. Wellen

4.1 Freie Wellenausbreitung im Vakuum



$\rho = 0$, $\underline{j} = 0$

$$\boxed{\begin{aligned}\Box\phi &= 0 \\ \Box\underline{A} &= 0\end{aligned}}$$

homog. Wellengln. (Lorentz-Eich.)

$$\underline{E} = -\dot{\underline{A}} - \underline{\nabla}\phi \quad , \quad \underline{B} = \underline{\nabla}\times\underline{A}$$

$$\boxed{\begin{aligned}\Box \underline{E} &= 0 \\ \Box \underline{B} &= 0\end{aligned}}$$

Allgem. Lösung von $\Box u(\underline{r},t) = 0$

$u(\underline{r},t) = F(\underline{k}\underline{r} - \omega t)$

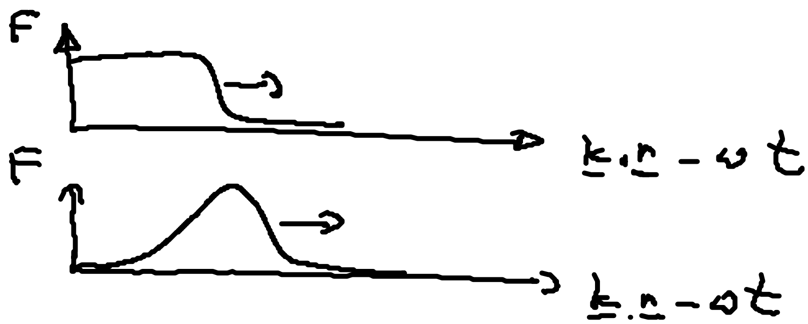
mit bel. 2x diff.baren Fkt. $F(\varphi)$

und $\boxed{\omega = c|\underline{k}|}$ d'Alembert'sche Lösung

Beweis: $\Box F = \left(\underline{k}^2 - \frac{\omega^2}{c^2}\right) F''(\varphi) = 0$

F muß nicht periodisch in φ sein,

z.B. solitäre Wellen:



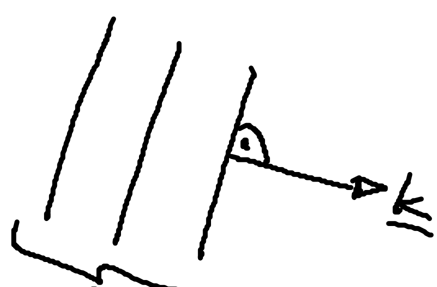
Front „kink"

Puls „Domäne"

$\underline{k}$ Wellenvektor

ω Frequenz

φ Phase



Flächen konstanter Phase

ebene Welle: $\underline{k}\cdot\underline{r} - \omega t = \varphi = \text{const}$

$\underline{r}(t) = \frac{1}{k^2}\underline{k}(\omega t + \varphi)$

$\Rightarrow$ Phasengeschw. $\underline{v}_{ph} = \left.\frac{d\underline{r}}{dt}\right|_{\varphi = \text{const.}} = \frac{\underline{k}}{k^2}\omega = c\underbrace{\frac{\underline{k}}{k}}_{=:\underline{n}}$