13.4. Pauli-Spinoren & Pauli-Gleichung

a) „Der Spin wirkt im Raum $R_{1/2}$":

- Spinzustände

$$\boxed{\begin{aligned} \left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle &= |+\rangle = |\uparrow\rangle \\ \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle &= |-\rangle = |\downarrow\rangle \end{aligned} \Bigg\} \text{VONS in } R_{1/2} \quad \begin{aligned} \langle \pm | \pm \rangle &= 1 \\ \langle + | - \rangle &= 0 \end{aligned}} \tag{13.17}$$

$j = \frac{1}{2}$ Spinquantenzahl

magn. Quantenzahl

$$\boxed{\begin{aligned} S^2 |\pm\rangle &= s(s+1)\hbar^2 |\pm\rangle = \tfrac{3}{4}\hbar^2 |\pm\rangle \\ S_z |\pm\rangle &= m_s \hbar\, |\pm\rangle = \pm\tfrac{1}{2}\hbar |\pm\rangle \end{aligned}} \tag{13.18}$$

$$S_{\pm} = S_x \pm i S_y \tag{13.19}$$

Kommutatoren (11.21) – (11.24)

allgemeinbedeutend: $|\eta\rangle = \alpha_+|+\rangle + \alpha_-|-\rangle$

i) $$\boxed{\underset{\sim}{\eta} = \begin{pmatrix} \langle +|\eta\rangle \\ \langle -|\eta\rangle \end{pmatrix} = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} \quad (13.21)$$

insbesondere: $$\boxed{\begin{aligned} |+\rangle &\rightarrow \underset{\sim}{\eta}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-\rangle &\rightarrow \underset{\sim}{\eta}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}} \quad (13.22)$$

Pauli-Spinor

ii) Spinoperatoren: (11.3)

$$S_i = \frac{\hbar}{2}\sigma_i \quad \text{mit } \sigma_i = \begin{pmatrix} \langle +|\sigma_i\, +\rangle & \cdots \\ \cdots & \cdots \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad (13.2)$$

σ_i : Paulische Spinmatrizen

iii) $$\left.\begin{array}{ll} \sigma_z \eta_+ = \eta_+ , & \sigma_x \eta_+ = \eta_- \\ \sigma_+ \eta_+ = 0 , & \sigma_y \eta_+ = i\eta_- \\ \sigma_- \eta_+ = 2\eta_- , & \sigma_z \eta_- = -\eta_- \\ \sigma_+ \eta_- = 2\eta_+ , & \sigma_- \eta_- = 0 \\ \sigma_x \eta_- = \eta_+ , & \sigma_y \eta_- = -i\eta_+ \end{array}\right\} \quad (13.24)$$

iv) Relationen der σ_i:

$$\boxed{\begin{array}{lll} (1) & \text{s. Gl. } (11.37) & \\ (2) & \mathrm{Sp}\, \sigma_i = 0 & (13.25) \\ (3) & \det \sigma_i = -1 & (13.26) \\ (4) & (\underline{\sigma} \cdot \underline{a})(\underline{\sigma} \cdot \underline{b}) = \underline{a} \cdot \underline{b}\, \mathbb{1} + i\underline{\sigma} \cdot (\underline{a} \times \underline{b}) & (13.27) \end{array}}$$

$i = 1 \ldots 3$

b) Produktraum $\mathcal{H} \otimes R_{1/2}$

$\hat{=}$ Raum der Spinorwellenfunktionen:

$$\boxed{|\chi\rangle = |\psi\rangle |\eta\rangle}$$

mit $|\psi\rangle \in \mathcal{H}$ und $|\eta\rangle \in R_{1/2}$

Skalarprodukt: $\langle \chi_1 | \chi_2 \rangle = \langle \psi_1 | \psi_2 \rangle \langle \eta_1 | \eta_2 \rangle$

diskretes VONS:

$$\boxed{\left.\begin{array}{ll} \{ \dots |\varphi_n\rangle \dots \} & \text{VONS in } \mathcal{H} \\ \{ |+\rangle, |-\rangle \} & \text{VONS in } \mathcal{R}_{1/2} \end{array}\right\} \to \begin{array}{l} \{ \dots |\varphi_n\rangle |+\rangle \dots \} \\ = \{ \dots |\varphi_n +\rangle, |\varphi_n -\rangle \} \end{array}}$$

VONS in $\mathcal{H} \otimes \mathcal{R}_{1/2}$

(13.30)

mit (1) Orthonormalität:

$$\langle \varphi_n \varepsilon | \varphi_m \varepsilon' \rangle = \langle \varphi_n | \varphi_m \rangle \langle \varepsilon | \varepsilon' \rangle = \delta_{nm} \delta_{\varepsilon\varepsilon'} \quad (13.31)$$

2 Entwicklungssatz:

$$|\chi\rangle = \sum_n \sum_{\varepsilon \pm} c_n(\varepsilon) |\varphi_n \varepsilon\rangle \quad (13.32)$$

mit $c_n(\varepsilon) = \langle \varphi_n \varepsilon | \chi \rangle$

Darstellung als Spinor im Ortsraum

$$\boxed{\begin{aligned} |\chi\rangle &= \int d^3r \left(\psi_+(\underline{r})\, |\underline{r}+\rangle + \psi_-(\underline{r})\, |\underline{r}-\rangle \right) \\ \text{mit} \quad \psi_\pm(\underline{r}) &= \langle \underline{r} \pm | \chi \rangle \qquad (13.33) \\ \rightarrow \text{allg. Pauli Spinor: } \underline{\chi} &= \begin{pmatrix} \psi_+(\underline{r}) \\ \psi_-(\underline{r}) \end{pmatrix} \end{aligned}}$$

c) Pauli-Gleichung:

$$\boxed{i\hbar \frac{d}{dt} |\chi\rangle = H|\chi\rangle = (H_0 + H_1 + H_{SB})|\chi\rangle}$$

(13.35)

mit Pauli-Spinor:

$$\boxed{i\hbar \frac{d}{dt} \begin{pmatrix} \psi_+(\underline{r}) \\ \psi_-(\underline{r}) \end{pmatrix} = H \begin{pmatrix} \psi_+(\underline{r}) \\ \psi_-(\underline{r}) \end{pmatrix}} \qquad (13.36)$$

13.5 Rabi-Gleichung

Motivation: i) Spinmanipulation mit B-Feld

ii) Modell-System für 2-Niveau-System

<u>Situation</u>: e^- - Spin in zeitabhän. Magnetfeld $\underline{B}(t)$
& <u>kein</u> Bahndrehimpuls $l = 0$

(13.16)
$$\boxed{H = H_0 + H_1 \text{ mit } H_0 = \frac{p^2}{2m} - e\varphi(\underline{r}) \qquad H_1(t) = \frac{e}{m}\underline{B}(t)\cdot\underline{S}}$$
(13.37)

$|\chi\rangle = |\psi\rangle|\eta\rangle$ einsetzen

$$i\hbar\frac{d}{dt}|\chi\rangle = i\hbar\left(\frac{d}{dt}|\psi\rangle\right)|\eta\rangle + i\hbar|\psi\rangle\left(\frac{d}{dt}|\eta\rangle\right)$$
$$= \left(H_0|\psi\rangle\right)|\eta\rangle + |\psi\rangle H_1|\eta\rangle$$

$\xrightarrow{\text{Entkopplung}}$: $i\hbar\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle$

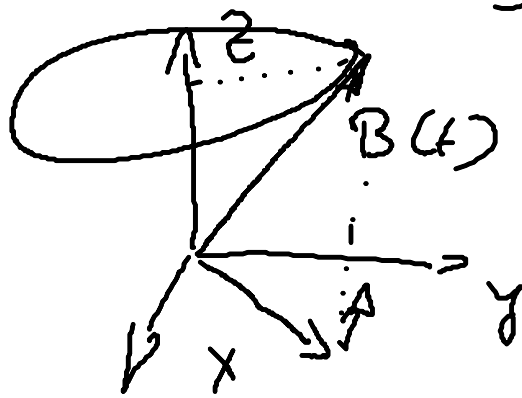
(13.38)
$$\boxed{i\hbar\frac{d}{dt}|\eta\rangle = H_1(t)|\eta\rangle}$$

Rabi-Gl.

<u>Methode der magn. Resonanz</u>:

= $\underline{B}$ - Feld induziert Spinumklappen

i) $\underline{B}(t) = B_0 \, \underline{e}_z + \underline{B}_1(t)$, $\underline{B}_1(t) = B_1 [\cos(\omega t)\, \underline{e}_x + \sin(\omega t)\, \underline{e}_y]$



ii) $H_1(t) = \frac{e}{m} \underline{B} \cdot \underline{S} = \omega_0 S_3 + \omega_1 \big(\cos(\omega t) S_1 + \sin(\omega t) S_2 \big)$ (13.39)

$\omega_0 = \frac{e B_0}{m}$

$\omega_1 = \frac{e B_1}{m}$

iii) Darstellung bz. $\{ |+\rangle, |-\rangle \}$, $\underline{S} = \frac{\hbar}{2} \underline{\sigma}$ (13.23)

$$\xrightarrow[\text{o.B.}]{} \quad H_1(t) = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & \omega_0 \end{pmatrix} \quad (13.40)$$

iv) Rabi-Gleichung: $|\psi\rangle \; \alpha_+(t)|+\rangle + \alpha_-(t)|-\rangle$

$$i\hbar \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = \frac{\hbar}{2} \underbrace{\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}}_{\tilde{H}} \begin{pmatrix} \beta_+(t) \\ \beta_-(t) \end{pmatrix}$$

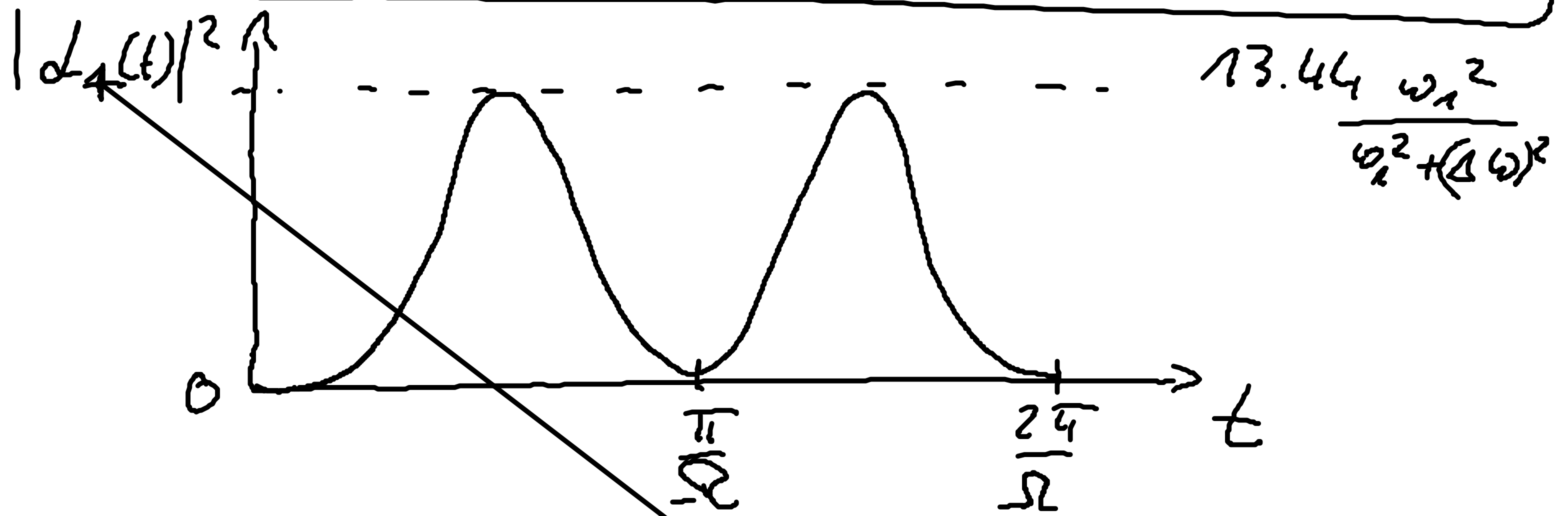
$\Delta\omega = \omega - \omega_0$ (13.42)

$\tilde{H}$ ... zeitunabh.

" Energie-EW " von $\tilde{H}$: $\tilde{E}_\pm = \pm\frac{\hbar}{2}\sqrt{(\Delta\omega)^2+\omega_1^2}$

(vi) Wahrscheinlichkeit, z.Z. t den Zustand $|+\rangle$ vorzufinden, wenn bei $t=0$ $|-\rangle$ vorlag? also $|\alpha_+(t)|^2$

$$\boxed{|\alpha_+(t)|^2 = \frac{\omega_1^2}{\omega_1^2+(\Delta\omega)^2}\sin^2\left[\frac{1}{2}\sqrt{(\Delta\omega)^2+\omega_1^2}\,t\right]}$$



$\Delta\omega = 0$, Resonanz

$\alpha_+ = 1$ für $\frac{2n+1}{2}\frac{\pi}{\Omega}$