

1.3 Bifurkationen

Klassifizierung

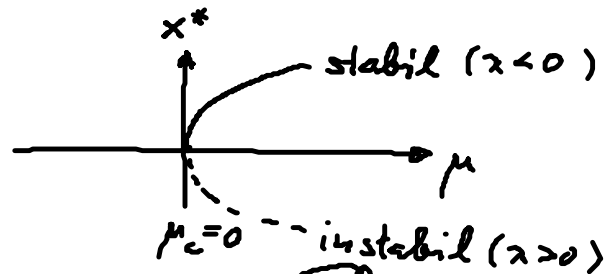
(A) Eigenwert-Null-Bifurkation

$$\lambda < 0 \rightarrow \lambda > 0$$

$$\det A > 0 \rightarrow \det A < 0$$

(A1) Sattel-Knoten-Bifurkation

$$\dot{x} = \mu - x^2$$



(A2) Transkrit. Bifurkation

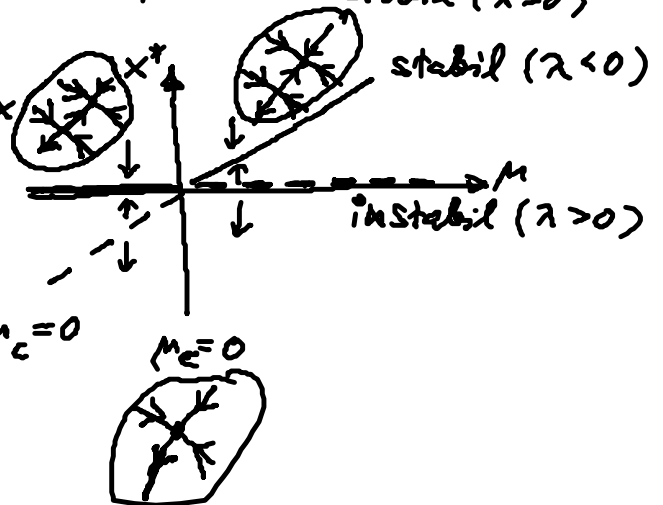
$$\dot{x} = \mu x - x^2$$

$$x^* = \begin{cases} 0 \\ \mu \end{cases}$$

$$\delta \dot{x} = (\mu - 2x^*) \delta x$$

$$\lambda = \begin{cases} \mu \\ -\mu \end{cases}$$

Stab.wechsel bei $\mu_c = 0$

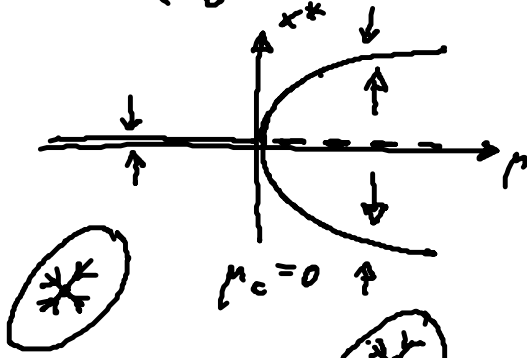


(A3) Stimmungabel-Bifurkation (pitchfork bifurcation)

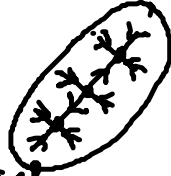
$$\dot{x} = \mu x - x^3$$

$$\delta \dot{x} = (\mu - 3x^{*2}) \delta x$$

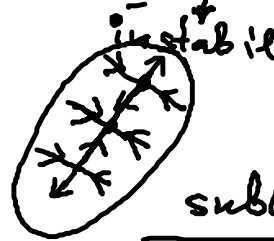
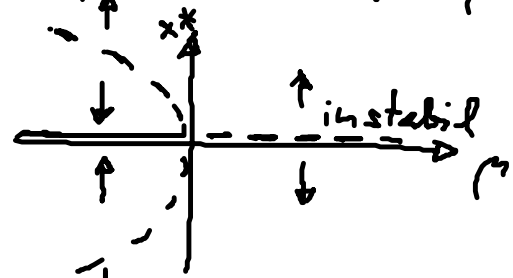
$$x^* = \begin{cases} \pm\sqrt{\mu} & (\text{für } \mu \geq 0) \\ 0 & \end{cases}$$



Supercritisch



$$a = \begin{cases} -2\mu & \text{stabil für } \mu > 0 \\ \mu & \text{stabil für } \mu < 0 \end{cases}$$



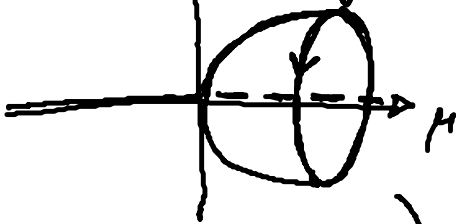
subkritisch

$$\dot{x} = \mu x + x^3$$

(B) Hopf-Bifurkation

$$\lambda_{1,2} = \alpha_0 \pm i\omega$$

α_0 st. Grenzzyklus



stab. Fokus



instab. Fokus + Grenzzyklus

(Andronov-Hopf) : Superkrit.

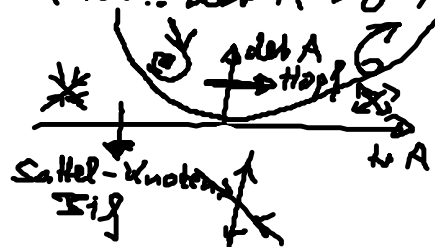
mit $\alpha_0 < 0 \rightarrow \alpha_0 > 0$

stabiler Fokus

instabiler Fokus + Grenzzyklus

$n=2$: $\text{tr } A < 0 \rightarrow \text{tr } A > 0$

(Vor.: $\det A > 0$)



mind. $n=2$ nötig!

Hopf-Normalform : generische Taylorentwicklung
in der Nähe des Hopf-Bif.
(2D-Zentrums-Mannigfaltigkeit)

$$\dot{z} = (\underbrace{\lambda + i\omega}_{\text{linearer Anteil}} \mp \underbrace{(1+i\gamma)|z|^2}_{\text{nichtlin. Anteil}}) z \quad z = x + iy \in \mathbb{C}$$

linearer Anteil nichtlin. Anteil \Rightarrow Bifurkation
eines Grenzzyklus
(limit cycle LC)
= period. Orbit

$$\delta \dot{z} = (\lambda + i\omega) \delta z$$

Fixpt. $z=0$ } $\lambda < 0$: stab. Fokus
Eigenwert $\lambda + i\omega$ } $\lambda > 0$: instab. Fokus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}}_A \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \Lambda = \frac{\text{tr} A}{2} \pm \sqrt{\left(\frac{\text{tr} A}{2}\right)^2 - \det A}$$

$$= \lambda \pm \sqrt{\lambda^2 - \lambda^2 - \omega^2}$$

$$= \lambda \pm i\omega$$

Transf. auf Amplitude r und Phase φ :

$$z(t) = r(t) e^{i\varphi(t)} \Rightarrow \dot{r} e^{i\varphi} + i\dot{\varphi} r e^{i\varphi} = (\lambda + i\omega \mp (1+i\gamma)r^2) r e^{i\varphi}$$

$$\text{Re: } \dot{r} = (\lambda \mp r^2) r \Rightarrow r=0 \text{ or } r^2 = \pm \lambda \quad (\lambda \geq 0)$$

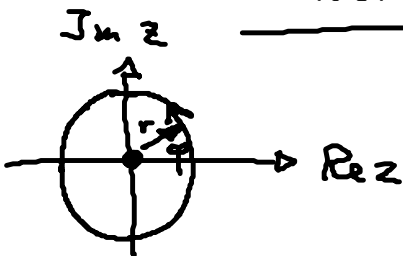
$$\text{Im: } \dot{\varphi} = (\omega \mp \gamma r^2) \Rightarrow \dot{\varphi} = \omega - \gamma \lambda \text{ Frequ.}$$

$$\Rightarrow \varphi = (\omega - \gamma \lambda) t$$

Lös. : $r=0$ (Fixpt.)

$$z(t) = \sqrt{\pm \lambda} e^{i(\omega - \gamma \lambda) t} \quad \text{für } \lambda \geq 0$$

Stuart-Landau-Oszillator



Im Bif. pt. ($\lambda=0$) :

$$\text{Amplitude } r = \sqrt{\pm \lambda} \rightarrow 0$$

$$\text{Frequenz } \omega \neq 0$$

lin. Stabilität des LC

im allgemeinen: Floquet-Theorie

$\dot{z} = f(z)$, period. Orbit $z^*(t) = z^*(t+T)$

$\delta \dot{z} = Df|_{z^*(t)} \delta z$, mit $Df(t) = Df(t+T)$ lin. ODE mit period. Koeff.

Lösung $\delta z(t) = \sum_j c_j e^{\Lambda_j t} u_j(t)$ mit $u_j(t) = u_j(t+T)$
Floquet-Exponenten $\Lambda_j \in \mathbb{C}$ Floquet-Mode

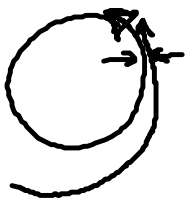
$\Rightarrow \Lambda u + \dot{u} = Df u$, $\delta z(t) = U(t) \delta z(0)$

Floquet-Multiplikatoren $\mu = e^{\Lambda T}$ (stabil, falls $|\mu| < 1$)
 = Eigenwerte von $U(T)$

hier: analyt. Lösung möglich in r, φ :

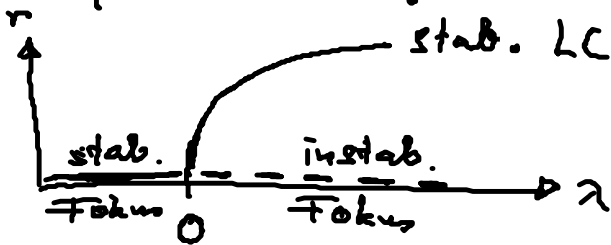
$\begin{pmatrix} \delta \dot{r} \\ \delta \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \lambda + 3r^2 & 0 \\ +2r\lambda & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta \varphi \end{pmatrix} = \underbrace{\begin{pmatrix} -2\lambda & 0 \\ +2r\sqrt{1+\lambda} & 0 \end{pmatrix}}_A \begin{pmatrix} \delta r \\ \delta \varphi \end{pmatrix}$

Floquet-Exponenten sind die Eigenwerte von A : $\Lambda^2 + 2\lambda \Lambda = 0$



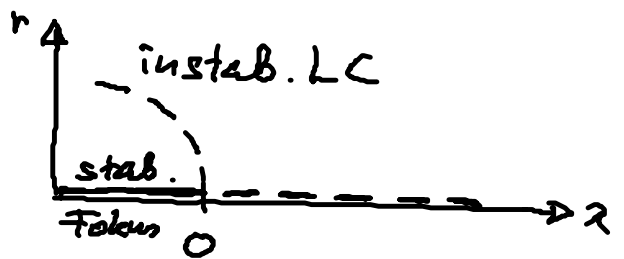
$\Rightarrow \Lambda = \begin{cases} 0 & \text{Goldstone-Mode} \\ -2\lambda & \leq 0 (\lambda \geq 0) \end{cases}$
 (transv. Floquet-Exp.)

Bifurkationsdiagramm:



Supercrit. Hopf-Bif.

$\dot{z} = (\lambda + i\omega - (1+ij)|z|^2)z$



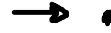
Subkrit. Hopf-Bif.

$\dot{z} = (\lambda + i\omega + (1+ij)|z|^2)z$

(C) Lokale Bij. von Grenzzyklen

(C1) Sattel-Knoten-Bij. eines Grenzzyklus (Kondensation v. Pfaden)

(C2) Perioden-Verdopplung (Flip)



(C3) Sekundäre Hopf-Bij. eines Grenzzyklus
(Sacker-Neimark)

