

Superconductivity continued

$$\phi(\underline{r}) = \sqrt{n_s} e^{i f(\underline{r})}$$

$$\underline{j}(\underline{r}) = \frac{ie^*}{2m^*} (\phi^* \nabla \phi - \phi \nabla \phi^*) - \frac{e^*}{m^*} \underline{A} \phi^* \phi = 0$$

$$\frac{ie^*}{2m^*} [i \nabla f \phi^* \phi + i \nabla f \phi \phi^*] - \frac{e^*}{m^*} \underline{A} \phi^* \phi = 0$$



$$\Rightarrow -\frac{e^*}{m^*} \phi^* \phi [\nabla f + e^* \underline{A}] = 0$$

$$\Rightarrow \underline{A}(\underline{r}) = -\frac{\nabla f(\underline{r})}{e^*} \quad (\text{the phase matters})$$

→ insert into flux equation

$$\Phi_B = -\oint_C d\underline{l} \cdot \frac{\nabla f}{e^*} = \frac{\delta f}{e^*}$$

↙ phase change for round trip round contour

δf is given modulo 2π

$$\Rightarrow \delta f = 2\pi \cdot \rho \quad \rho = 0, 1, 2, \dots$$

$$\Rightarrow \Phi_B = \frac{2\pi \hbar c}{e^*} \rho$$

$e^* = 2e$ found experimentally by Deaver and Fairbank in 1961

So we know now:

bosons with charge $2e$ give superconductivity

Ogg pairs

- in 1946 Ogg proposed electron pairing in real-space to yield bosons with $S=0$ or $S=1$
- later further developed by Schafroth, Butler and Blatt, but
 - no quantitative description of SC ($T_c \sim 10^4$ K much too large)
 - no microscopic force to explain attraction (electrons should repel each other)

\Rightarrow real space pairing does not work \checkmark
theory forgotten

Cooper pairs

- 1956 idea behind weak attraction by Cooper
 - \hookrightarrow consider Fermi liquid (e.g. free electron metal)
 - two quasiparticles with \underline{k} and $-\underline{k}$

two particle wave function

$$\Psi_0(\underline{r}_1, \underline{r}_2) = \sum_{\underline{k}} g_{\underline{k}} e^{i\underline{k}\cdot\underline{r}_1} e^{-i\underline{k}\cdot\underline{r}_2}$$

• assume spin singlet $\frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow)$ $\underline{S} = 0$

(triplet is also possible
 \Rightarrow see unconventional superconductors)

ψ_0 symmetric \Leftrightarrow antisymmetric

$$\psi_0(\underline{r}_1, \underline{r}_2) = \sum_{k > k_F} g_{\underline{k}} \cos \underline{k} \cdot (\underline{r}_1 - \underline{r}_2)$$

\Downarrow
 into two particle SE

$$\left[\sum_{i=1,2} \frac{p_i^2}{2m} + V(\underline{r}_1, \underline{r}_2) \right] \psi_0 = \bar{E} \psi_0$$

we get

$$(\bar{E} - 2E_{\underline{k}}) g_{\underline{k}} = \sum_{k' > k_F} \underline{V}_{\underline{k}\underline{k}'} g_{\underline{k}'}$$

\uparrow
 unperturbed
 plane-wave
 energies

\Downarrow Fourier transform of V

$$\underline{V}_{\underline{k}\underline{k}'} = \frac{1}{\Omega} \int_{\text{volume}} d^3r V(\underline{r}) e^{i(\underline{k}' - \underline{k}) \cdot \underline{r}}$$

\uparrow
 $\underline{r}_1 - \underline{r}_2$

since $E_{\underline{k}} > E_F$

if $\bar{E} < 2E_F$ we could have a bound state

question? does a set of $g_{\underline{k}}$ exist such that $\bar{E} < 2E_F$?

Coulomb potential:

$$V_{\underline{k}, \underline{k}'} = V_{q = \underline{k} - \underline{k}'} = \frac{4\pi e^2}{q^2} > 0 \quad \text{not attractive}$$

however: quasiparticles are no free electrons

⇒ Screening

↳ Thomas-Fermi for simplicity

$$\epsilon(\underline{k}) = \frac{\underline{k}^2 + \underline{k}_0^2}{\underline{k}^2} \neq 1$$

dielectric function

$$V(\underline{r}) = \frac{\epsilon^{-1}(\underline{r}_1 - \underline{r}_2)}{|\underline{r}_1 - \underline{r}_2|}$$

$$\Rightarrow V(q) = \frac{4\pi e^2}{q^2 + \underline{k}_0^2} \quad \text{still positive, but reduced}$$

build on this idea: ions are positively charged

⇒ they respond to motion of electrons, too
(although on timescale of phonons)

rough estimate (see Ashcroft Chapter 26)

$$V(q) = \frac{4\pi e^2}{q^2 + \underline{k}_0^2} \frac{\omega^2}{\omega^2 - \omega_q^2}$$

$$\omega = \frac{1}{\hbar} (\epsilon_{\underline{k}} - \epsilon_{\underline{k}'})$$

ω_q : typical phonon frequency at q

if $\omega < \omega_q$ this (oversimplified) interaction

could work ✓

\Rightarrow phonons could provide attraction

Cooper further simplified the potential

average phonon energy $\sim \hbar\omega_c$

$$V_{\underline{k}\underline{k}'} = \begin{cases} -V & \text{for } |\epsilon_{\underline{k}} - \epsilon_{\underline{k}_F}|, |\epsilon_{\underline{k}'} - \epsilon_{\underline{k}'_F}| < \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow g_{\underline{k}} = V \frac{\sum g_{\underline{k}'}}{2\epsilon_{\underline{k}} - \bar{\epsilon}}$$

apply $\bar{\epsilon}$ and then cancel $\sum g_{\underline{k}}$

$$\Rightarrow \frac{1}{V} = \sum_{\underline{k} > \underline{k}_F} \frac{1}{2\epsilon_{\underline{k}} - \bar{\epsilon}}$$

replace sum by integral

$$N(\epsilon) \approx N(\epsilon_F)$$

$$\frac{1}{V} = \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_c} d\epsilon N(\epsilon) \frac{1}{2\epsilon - \bar{\epsilon}} = \frac{1}{2} N(\epsilon_F) \ln \frac{2\epsilon_F - \bar{\epsilon} + 2\hbar\omega_c}{2\epsilon_F - \bar{\epsilon}}$$

$$\bar{\epsilon} \left(e^{-\frac{2}{N(\epsilon_F)V}} - 1 \right) = 2\epsilon_F \left(e^{-\frac{2}{N(\epsilon_F)V}} - 1 \right) + 2\hbar\omega_c e^{-\frac{2}{N(\epsilon_F)V}} \quad \leftarrow N(\epsilon_F)$$

weak coupling⁴: $N(\epsilon_F)V \ll 1 \Rightarrow 1 - e^{-\frac{2}{N(\epsilon_F)V}} \approx 1$

$$\bar{\epsilon} \approx 2\epsilon_F - 2\hbar\omega_c e^{-\frac{2}{N(\epsilon_F)V}} < 2\epsilon_F$$

energy for two independent electrons

\Rightarrow electron pair is stable

important:

- binding outweighs kinetic energy gain for states above ϵ_F , regardless of size of \checkmark
- binding energy is not analytic at $V=0$
 - \Rightarrow perturbation theory does not apply
 - \Rightarrow that's why theory took so long to develop
- binding happens in \underline{k} -space, not in real-space (like Ogg tried it)

$$\psi_0 = \sum_{k > k_F} g_k \cos k(\underline{r}_1 - \underline{r}_2)$$

$$= \sum_{k > k_F} \frac{\checkmark}{\lambda(\epsilon_k - \epsilon_F) + 2\epsilon_F - E} \cos k(\underline{r}_1 - \underline{r}_2)$$

Microscopic theory: BCS theory

we know our bosons, now find quantum mechanical description (ie. wavefunction, energy, etc.)

$$\Psi = \phi(\underline{r}_1, \sigma_1, \underline{r}_2, \sigma_2) \dots \phi(\underline{r}_{N-1}, \sigma_{N-1}, \underline{r}_N, \sigma_N)$$

\uparrow boson wfc, $\vec{\sigma}$ always the same

$$\Psi_{BCS} = A \Psi$$

\mathbb{R} antisymmetrize

put this into many-body SE and solve, maybe not too insightful
instead: the fast route: 2nd quantization

$$|H| = \sum_{\underline{k}, \sigma} \xi_{\underline{k}} c_{\underline{k}\sigma}^{\dagger} c_{\underline{k}\sigma} + \sum_{\underline{k}, \underline{k}'} c_{\underline{k}\uparrow}^{\dagger} c_{-\underline{k}'\downarrow}^{\dagger} V_{\underline{k}\underline{k}'} c_{-\underline{k}'\downarrow} c_{\underline{k}\uparrow}$$

\uparrow $\epsilon_{\underline{k}} - \epsilon_F$ \uparrow creation op. \uparrow annihilation op.

then find ground state, so on so forth

• instead: mean field approximation

$$\sum_{\underline{k}'} V_{\underline{k}\underline{k}'} c_{-\underline{k}'\downarrow} c_{\underline{k}\uparrow} \approx -V \Theta(\hbar\omega_c - |\xi_{\underline{k}}|)$$

$$\sum_{\underline{k}'} \Theta(\hbar\omega_c - |\xi_{\underline{k}'}|) \overline{(c_{-\underline{k}'\downarrow} c_{\underline{k}'\uparrow})_{\text{avg}}}$$

$$\Rightarrow -\Delta_{\underline{k}} = \begin{cases} -A & \text{for } |\xi_{\underline{k}}| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

one more step:

we write $-c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} = -c_{\underline{k}\downarrow} c_{\underline{k}\uparrow} + b_{\underline{k}} - b_{\underline{k}}$

\downarrow
 $\langle c_{-\underline{k}\downarrow} c_{\underline{k}\uparrow} \rangle_{\text{avg}}$

- insert into BCS Hamiltonian
- expand to first order
- apply mean-field approx. for V

$$\Rightarrow H_{MF} = \sum_{\underline{k}\sigma} \xi_{\underline{k}} c_{\underline{k}\sigma}^{\dagger} c_{\underline{k}\sigma} - \sum_{\underline{k}} \left[\Delta_{\underline{k}} c_{\underline{k}\uparrow}^{\dagger} c_{-\underline{k}\downarrow}^{\dagger} + \Delta_{\underline{k}}^{\dagger} c_{-\underline{k}\downarrow} c_{\underline{k}\uparrow} - \Delta_{\underline{k}} b_{\underline{k}} \right]$$

This is an effective single-pair approximation
 (i.e. only 2 c-operators used)

diagonalize MF Hamiltonian

• define new quasiparticles:

$$C_{\underline{k}\uparrow} = u_{\underline{k}} \alpha_{\underline{k}} + v_{\underline{k}} \beta_{\underline{k}}^{\dagger}$$

$\alpha_{\underline{k}}$ and $\beta_{\underline{k}}$ are new

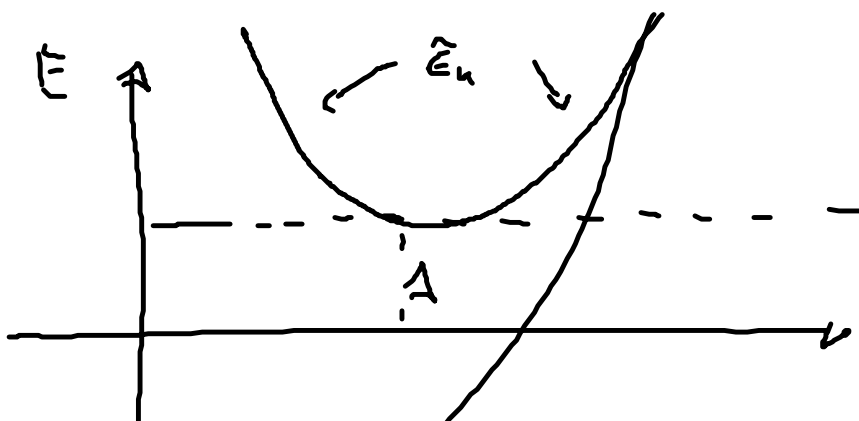
$$C_{\underline{k}\downarrow} = u_{\underline{k}} \beta_{\underline{k}} - v_{\underline{k}} \alpha_{\underline{k}}^{\dagger}$$

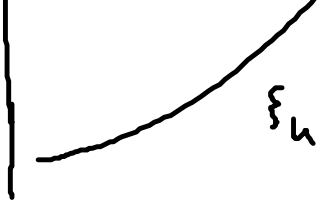
(rotated) quasiparticles

$$u_{\underline{k}}^2 = \frac{1}{2} \left(1 + \frac{\xi_{\underline{k}}}{\tilde{\epsilon}_{\underline{k}}} \right) \quad v_{\underline{k}}^2 = \frac{1}{2} \left(1 - \frac{\xi_{\underline{k}}}{\tilde{\epsilon}_{\underline{k}}} \right)$$

$$u_{\underline{k}} v_{\underline{k}} = -\frac{\Delta_{\underline{k}}}{2\tilde{\epsilon}_{\underline{k}}} \quad \text{with} \quad \tilde{\epsilon}_{\underline{k}} = \sqrt{\xi_{\underline{k}}^2 + |\Delta_{\underline{k}}|^2}$$

$$H_{MF} = \bar{E}_0 + \sum_{\underline{k}} \tilde{\epsilon}_{\underline{k}} (\alpha_{\underline{k}}^{\dagger} \alpha_{\underline{k}} + \beta_{\underline{k}}^{\dagger} \beta_{\underline{k}})$$





Other consequence of energy gap

Einstein like heat capacity

