

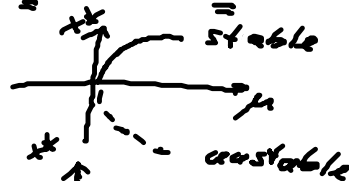
# English Summary: 1.3 Bifurcations

(A) bifurcations with eigenvalue zero:  $\lambda < 0 \Leftrightarrow \lambda > 0$

A: Jacobian at fixed point  $\underline{x}^*$   $\Rightarrow \det J > 0 \Leftrightarrow \det J < 0$

dimension  $n=1$

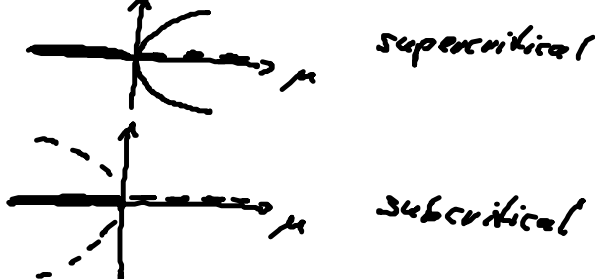
(A1) saddle-node bifurcation:  $\dot{x} = \mu - x^2$



(A2) transcritical bifurcation:  $\dot{x} = \mu x - x^2$



(A3) pitch fork bifurcation:  $\dot{x} = \mu x - x^3$



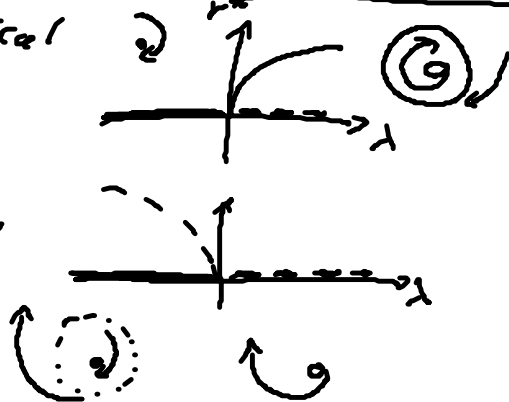
$$\dot{x} = \mu x + x^3$$

(B) Hopf bifurcation:  $\lambda_{1,2} = \lambda_0 \pm i\omega$ ,  $\lambda_0 < 0 \Leftrightarrow \lambda_0 > 0$   
 $z = x + iy = r e^{i\varphi} \in \mathbb{C}$

dimension  $n=2$

$$\dot{z} = (\lambda + i\omega \mp (1+i\gamma)|z|^2)z$$

supercritical  
 subcritical



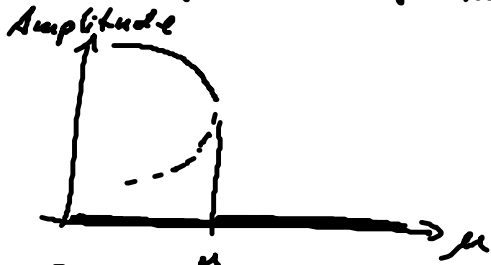
## (C) Lokale Bifurkationen eines Grenzzyklus

Startpunkt: Grenzzyklus, nicht Fixpunkt

=> Keine einfache lineare Stabilitätsanalyse

(C1) Sattel-Knoten-Bifurkation eines Grenzzyklus:  $n=2$

(fold bifurcation of a limit cycle, LC)

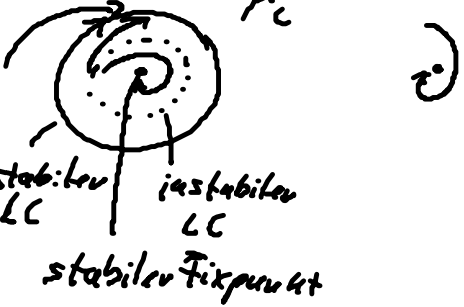
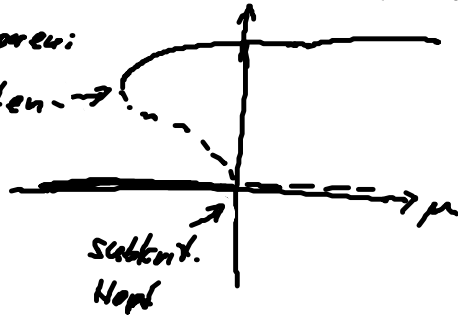


Amplitude  $\neq 0$

Frequenz  $\neq 0$

häufig in Kombination mit subkritischer Hopf-Bifurkation:

Sattel-Knoten-Bif.

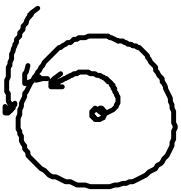


stabiler Fixpunkt

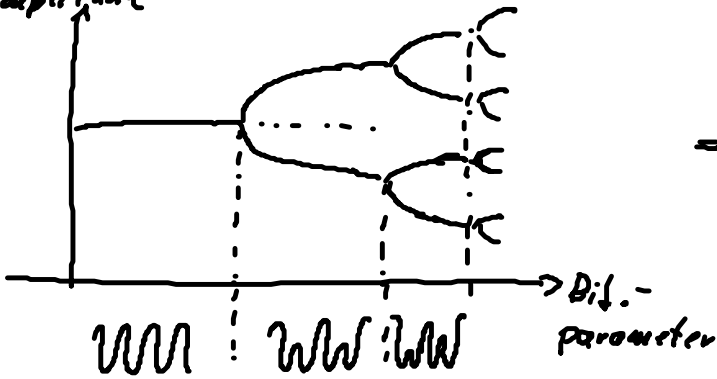
(C2) Periodenverdopplung (flip bifurcation)  $n=3$

o Möbiustrand

phase slip  $\pi$   
nach 1 Umlauf  
=> Torsion



maximale Amplitude



Floquet-Exponent:  $\Lambda = \lambda + i\omega$

Bifurkation:  $\lambda = 0, \omega T = \pi$

=> Floquet-Multiplikator:  $\mu = e^{\Lambda T} = e^{i\pi} = -1$   
Bifurkation

häufig als Periodenverdopplungskaskade  
=> Chaos (Feigenbaum-Szenario)

unendlich viele instabile periodische Orbits der Perioden  $2^k T$  ( $k=0,1,2,\dots$ )

Perioden:  $T \quad 2T \quad 4T \quad \dots \quad 2^k T$

(C3) Neimark-Sacker-Bifurkation:  $n=3$

(sekundäre Hopf-Bifurkation eines Grenzzyklus)



1. Hopf-Bifurkation des Fixpunktes:  $\phi_1 = \omega_1 t$

2. Hopf-Bifurkation des LC:  $\phi_2 = \omega_2 t$

inkommensurable Frequenzen:  $\omega_1/\omega_2 \notin \mathbb{Q}$  (quasiperiodisch)

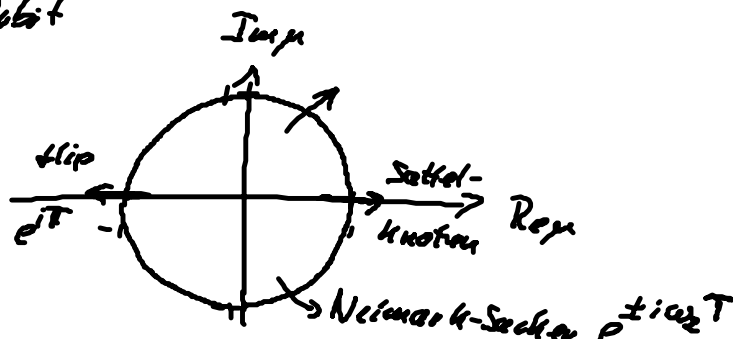
$\Rightarrow$  Trajektorie schließt sich nicht, liegt dicht auf dem Torus

falls  $\frac{\omega_1}{\omega_2} \in \mathbb{Q} \Rightarrow$  geschlossenes Orbit

Zusammenfassung C1-C3:

Stabilitätsgränze:

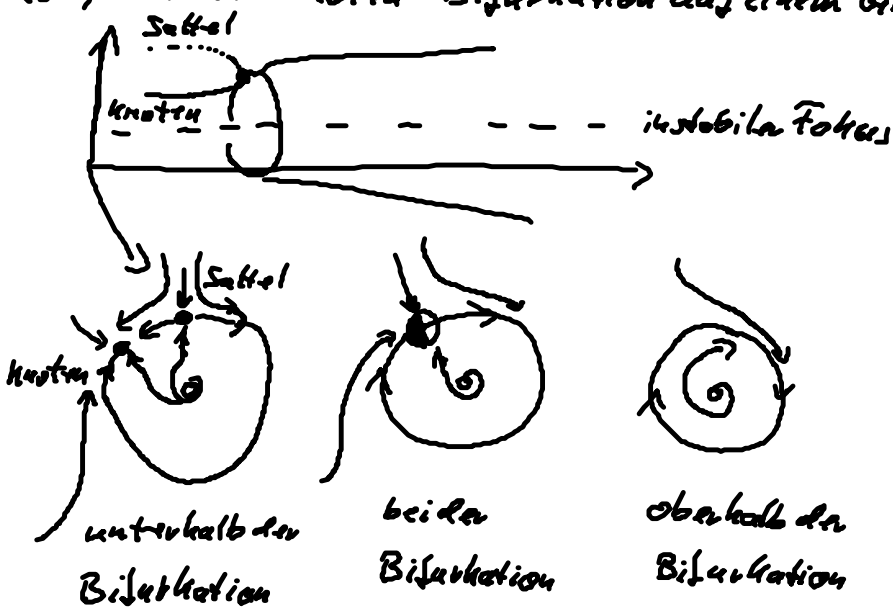
$$\text{Re } \lambda = 0 \Leftrightarrow |\mu| = 1$$



### (D) Globale Bifurkation von Grenzzyklen

globale qualitative Änderung des Phasenportraits

### (D1) Sattel-Knoten-Bifurkation auf einem Grenzzyklus (Onset explosion)



Amplitude  $\neq 0$

Frequenz  $\rightarrow 0$

(Periode  $\rightarrow \infty$ )

$\Rightarrow$  Saddle-node infinite period (SNIPER)

Saddle-node on invariant cycle (SNIC)

einfaches gezeigtes Modell für SNIPER

(Ditzinger et al.: PRE 50, 3508 (1994), Hu et al. PRL 71, 87 (1993),

Hizaidis, Aust, Schöll: Int. J. Bif. Chaos 18, 1759 (2008)

Aust, Hövel, Hizaidis, Schöll: Eur. Phys. J. S.T. 17, 77 (2010)

$$\left. \begin{aligned} \dot{x} &= x(1-x^2-y^2) + y(x-b) \\ \dot{y} &= y(1-x^2-y^2) - y(x-b) \end{aligned} \right\} \begin{aligned} &\text{in Polarkoordinaten} \\ &x = r \cos \varphi, y = r \sin \varphi \end{aligned} \quad \begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\varphi} &= b - r \cos \varphi \end{aligned}$$

# Modell der neuronalen Anregbarkeit von Typ I

⇒ S. Kapitel 7, "Neurodynamik"

## (E) Bifurkationen räumlicher Muster

$$\frac{\partial}{\partial t} \underline{q}(\underline{x}, t) = \underline{f}(\underline{q}, \mu) + D \Delta \underline{q} \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \quad \text{Laplace-Operator}$$

Ansatz:  $\delta \underline{q} \sim e^{i \underline{k} \cdot \underline{x}} e^{\lambda t} \xrightarrow{\text{Linearisierung}} A_{ij} = \frac{\partial f_i}{\partial q_j}(\underline{q}^*) - D_{ij} k^2$

⇒ Dispersionsrelation:  $\lambda(\underline{k})$

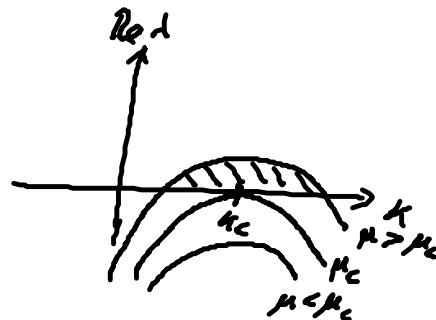
- ⇒  $\text{Re } \lambda(\underline{k}) < 0$  stabil
- $\text{Re } \lambda(\underline{k}) = 0$  Bifurkation
- $\text{Re } \lambda(\underline{k}) > 0$  instabil

### (E1) Turing-Instabilität:

$$\lambda(\underline{k}) = 0 \quad |\underline{k}| = k_c \neq 0$$

statische, räumlich periodische Strukturen

$$\lambda(\underline{k}) = \epsilon - b(k^2 - k_c^2)^2$$



### (E2) stehende/laufende Wellen:

$$\lambda(\underline{k}) = \pm i\omega \quad \text{bei } |\underline{k}| = k_c \neq 0$$

oszillierende, räumlich periodische Strukturen

$$\Rightarrow \text{Wellen: } \delta \underline{q} \sim e^{i(\underline{k} \cdot \underline{x} \pm \omega t)}$$

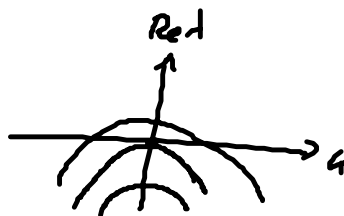
$$\lambda(\underline{k}) = \epsilon - b(k^2 - k_c^2)^2 \pm i\omega$$

### (E3) Hopf-Bifurkation:

$$\lambda = \pm i\omega \quad \text{bei } |\underline{k}| = k_c = 0$$

$$\lambda(\underline{k}) = \epsilon - b(k^2)^2 \pm i\omega$$

räumlich homogenen Grenzzyklus



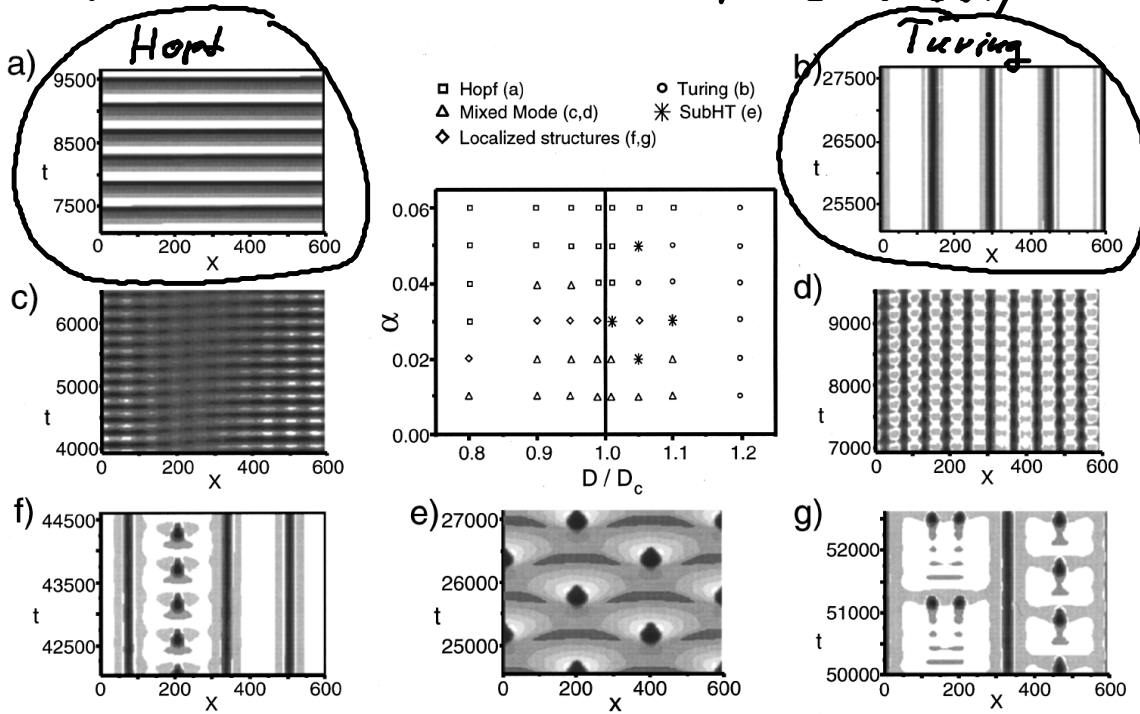


FIG. 5. Regimes of different asymptotic spatiotemporal behavior near the CTHP bifurcation given by the line  $D/D_c=1$ . The symbols in the  $(\alpha, D/D_c)$  control parameter space denote various types of space-time patterns which are illustrated by typical space-time plots of  $j(x,t)$  as insets: (a) Hopf oscillations (squares), (b) Turing patterns (dots), (c) and (d) Turing-Hopf mixed modes (triangles), (e) subharmonic Turing-Hopf mode consisting of spatiotemporal spiking (asterisks), and (f) and (g) localized Turing-Hopf structures (diamonds). (For parameters, see Table I.)

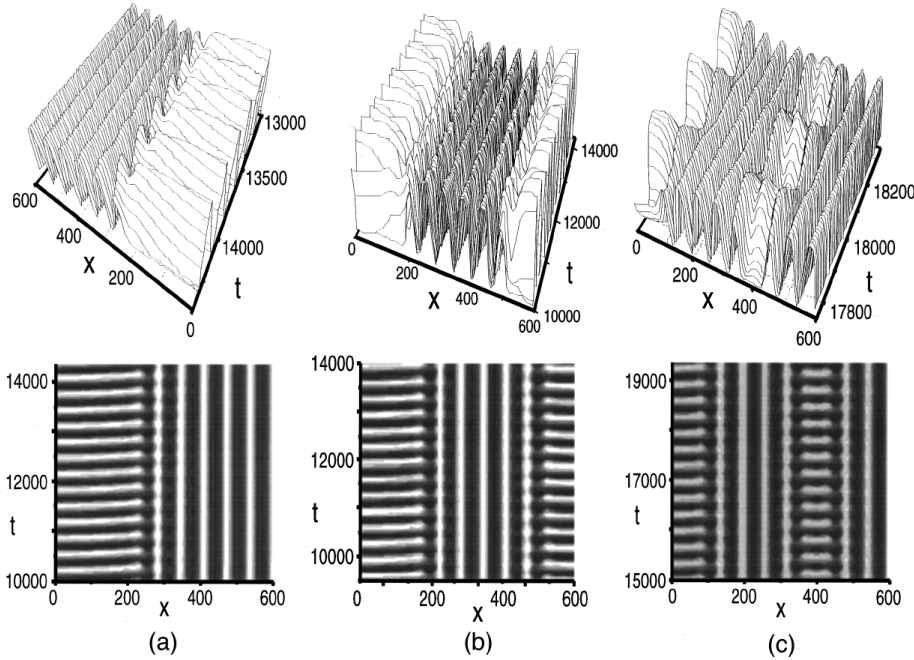


FIG. 6. Localized structures near the CTHP  $C_2$  for  $T=0.05$ ,  $\alpha=0.02$ ,  $D=8$  (corresponding to  $D/D_c=0.67$ ),  $j_0=3.1$ , and different initial conditions: (a) Turing-Hopf front. (b) Turing domain embedded between two Hopf states. (c) Localized Turing-Hopf structures. The current density  $j(x,t)$  is shown as a density plot and as a three-dimensional representation.