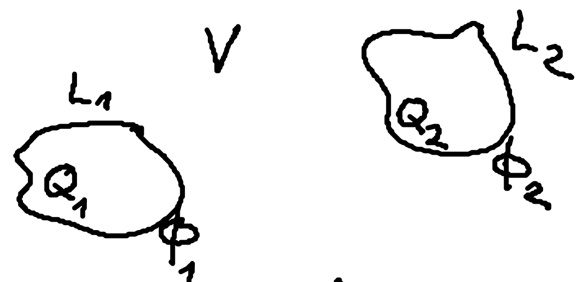
2. Grundaufgabe:

geg.: Leiter L_α (Oberfläche S_α), $\alpha = 1, \dots, n$
mit Ladungen Q_α;
Raumladungsdichte $\rho(\underline{r})$ im Außenraum V

gesucht: $\phi(\underline{r})$; ϕ_α



Lösung: Zurückführen
auf die 1. Grundaufgabe
durch Zusammenhang zw.

Es gilt:
$$\boxed{Q_\alpha = \sum_{\beta=1}^{n} C_{\alpha\beta}\, \phi_\beta} \qquad \alpha = 1, \dots, n$$

mit Kapazitätskoeffizienten $C_{\alpha\beta}$

Beweis:
$$Q_\alpha = -\epsilon_0 \oint_{S_\alpha} d\underline{f} \cdot \underline{\nabla}\phi(\underline{r})$$
$$= -\epsilon_0 \oint_{S_\alpha} d\underline{f} \cdot \underline{\nabla}_r \int_V d^3r'\, G(\underline{r} - \underline{r}')\, \rho(\underline{r}')$$

$$- \epsilon_0^2 \oint_{S_\alpha} d\underline{f} \cdot \underline{\nabla}_r \sum_\beta \phi_\beta \oint_{S_\beta} d\underline{f}' \cdot \underline{\nabla}_{r'} G(\underline{r}-\underline{r}')$$

$$= - \epsilon_0 \int_{L_\alpha} d^3r \int_V d^3r' \underbrace{\Delta_r G(\underline{r}-\underline{r}')}_{-\frac{1}{\epsilon_0}\delta(\underline{r}-\underline{r}') = 0 \text{ für } \underline{r} \in L_\alpha,\ \underline{r}' \in V} \rho(\underline{r}')$$

$$- \sum_{\beta=1}^{n} \phi_\beta \underbrace{\epsilon_0^2 \oint_{S_\alpha} d\underline{f} \cdot \underline{\nabla}_r \oint_{S_\beta} d\underline{f}' \cdot \underline{\nabla}_{r'} G(\underline{r}-\underline{r}')}_{=: -C_{\alpha\beta}}$$

$$= \sum_{\beta=1}^{n} C_{\alpha\beta} \phi_\beta \qquad \square$$

Symmetrie $G(\underline{r}-\underline{r}') = G(\underline{r}'-\underline{r})$

$$\Rightarrow \quad \boxed{C_{\alpha\beta} = C_{\beta\alpha}}$$

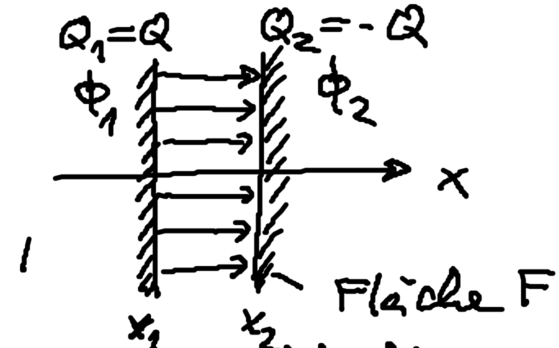
Einheit der Kapazität: $1\,F = 1 \frac{C}{V} = 1$ Farad

(M. Faraday 1791-1867)

Betrachte speziell 1 Leiter mit Pot. ϕ_L:

$$C = \frac{Q}{\phi_L} \quad \underline{\text{Kapazität des Leiter}}$$

<u>Beispiel</u>: Plattenkondensator



$$\left.\begin{aligned} Q_1 &= C_{11}\phi_1 + C_{12}\phi_2 \\ Q_2 &= C_{21}\phi_1 + C_{22}\phi_2 \end{aligned}\right\} C_{12} = C_{21} \equiv C'$$

Symm. $1 \Leftrightarrow 2 \quad \Rightarrow C_{11} = C_{22} \equiv C$

Spezialfall: $Q_1 + Q_2 = 0$

$$\Rightarrow \quad \begin{aligned} Q &= C\phi_1 + C'\phi_2 \\ -Q &= C'\phi_1 + C\phi_2 \\ \hline 0 &= (C + C')(\phi_1 + \phi_2) \end{aligned} \quad \Rightarrow C = -C'$$

$$C = -C' = \frac{Q}{\phi_1 - \phi_2} \qquad (1)$$

E-Feld fast nur zwischen den Platten

$$\Rightarrow \quad \sigma = \frac{Q}{F} = \epsilon_0 E = \text{const.} \qquad (2)$$

$$\Rightarrow \phi(x) = -Ex + \phi_0$$

$$\Rightarrow \phi_1 - \phi_2 = E(x_2 - x_1) = Ea \qquad (3)$$

$$(1), (2), (3) \Rightarrow \quad \boxed{C = -C' \overset{(3)}{=} \frac{Q}{Ea} = \epsilon_0 \frac{F}{a}}$$

Lösung der 2. Grundaufgabe:

<u>Inverse der Kapazitätsmatrix</u>

$$\phi_\alpha = \sum_{\beta=1}^{n} (C^{-1})_{\alpha\beta}\, Q_\beta$$

eingesetzt in die Lösung der 1. Grundaufgabe

$\Rightarrow \phi(\underline{r})$ für geg. Q_β, $\rho(\underline{r})$. $\square$

Energie des Feldes im Außenraum ($\rho(\underline{r})=0$)

$$W = \frac{\epsilon_0}{2}\int_V d^3r\,\left(\underline{E}(\underline{r})\right)^2$$

diff. Änderung $\quad Q_\alpha \longrightarrow Q_\alpha + \delta Q_\alpha$

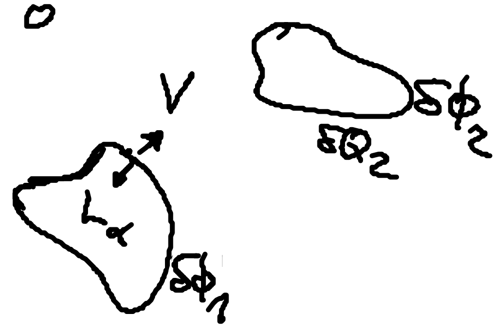
$$\phi_\alpha \longrightarrow \phi_\alpha + \delta\phi_\alpha$$

Lösung $\quad \phi \longrightarrow \phi(\underline{r}) + \delta\phi(\underline{r})$

$$\delta W = \frac{\epsilon_0}{2}\int d^3r\; 2\,\underbrace{\underline{E}(\underline{r})}\,\delta\underline{E}(\underline{r})$$

$$= -\epsilon_0 \int_V d^3r\; \underbrace{\left(\underline{\nabla}\phi(\underline{r})\right)\delta\underline{E}(\underline{r})}_{\underline{\nabla}(\phi\,\delta\underline{E}) - \phi\,\underbrace{\underline{\nabla}\delta\underline{E}}_{0}}$$

$$\underline{\nabla}\cdot\delta\underline{E} = \delta\,\underbrace{\underline{\nabla}\cdot\underline{E}}_{0} = 0 \quad \text{in } V$$



$$\delta W = -\epsilon_0 \int_V d^3r\; \underline{\nabla}(\phi\,\delta\underline{E})$$

$$\overset{\text{Gauß}}{=} \epsilon_0 \sum_\alpha \oint_{S_\alpha} d\underline{f}\,\left(\phi(\underline{r})\,\delta\underline{E}(\underline{r})\right)$$

$$\overset{\phi(\underline{r})|_{S_\alpha}=\phi_\alpha}{=} \epsilon_0 \sum_\alpha \phi_\alpha \oint_{S_\alpha} d\underline{f}\cdot\delta\underline{E}$$

$$= \sum_\alpha \phi_\alpha\,\delta Q_\alpha \qquad\qquad \delta Q_\alpha = \sum_\beta C_{\alpha\beta}\,\delta\phi_\beta$$

$$= \sum_{\alpha,\beta} \phi_\alpha\, C_{\alpha\beta}\,\delta\phi_\beta$$

$$= \frac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\,\phi_\alpha\,\delta\phi_\beta + \frac{1}{2}\sum_{\alpha\beta} \underbrace{C_{\beta\alpha}}_{C_{\alpha\beta}}\,\phi_\beta\,\delta\phi_\alpha$$

$$= \frac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\,\underbrace{\left(\phi_\alpha\,\delta\phi_\beta + \phi_\beta\,\delta\phi_\alpha\right)}_{\delta(\phi_\alpha\phi_\beta)}$$

$$\delta W = \delta\left\{\frac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\,\phi_\alpha\,\phi_\beta\right\}$$

$$\Rightarrow \quad \boxed{W = \frac{1}{2}\sum_{\alpha\beta} C_{\alpha\beta}\,\phi_\alpha\,\phi_\beta} \qquad \underline{\text{Feldenergie}}$$

2. Stationäre Ströme und Magnetfeld

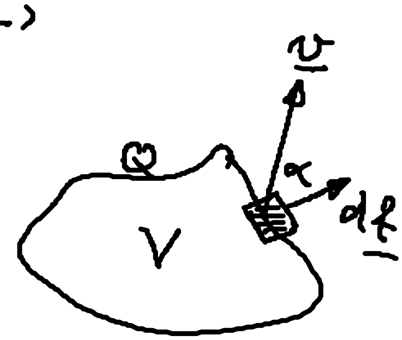
2.1. Kontinuitätsgleichung

Bewegte Ladungen $\longrightarrow$ el. Strom I

<u>Exp. Erfahrung</u>: Ladungserhaltung

$$Q(t) = \int_V d^3r\,\rho(\underline{r},t)$$

$\Rightarrow$ <u>globaler</u> Erhaltungssatz



$$\frac{dQ}{dt} \equiv \frac{d}{dt}\int_V d^3r\,\rho(\underline{r},t) = -\oint_{\partial V} \delta I$$

$$\delta I = \frac{\rho\,dV}{dt} = \rho\,\frac{|\underline{v}|\,dt\,|d\underline{f}|\cos\alpha}{dt} = \rho\,\underline{v}\cdot d\underline{f}$$

El. Stromdichte $\underline{j}(\underline{r},t) := \rho(\underline{r},t)\,\underline{v}(\underline{r},t)$

$$\frac{d}{dt}\int_V d^3r\,\rho(\underline{r},t) = -\oint_{\partial V} d\underline{f}\cdot\underline{j} \overset{\text{gauß}}{=} -\int_V d^3r\,\operatorname{div}\underline{j}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\rho(\underline{r},t) + \operatorname{div}\underline{j} = 0}$$ Kontin.-gl. (lokaler Erhalt.satz)

stat. Lad.verteil $\Rightarrow \operatorname{div}\underline{j} = 0$

2.2. Magn. Induktion

Exp. Erfahrung

El.statik

Kraft $\boxed{\underline{F} = q\,\underline{v}\times\underline{B}(\underline{r})}$ Lorentz-Kraft $\underline{F} = q\,\underline{E}(\underline{r})$

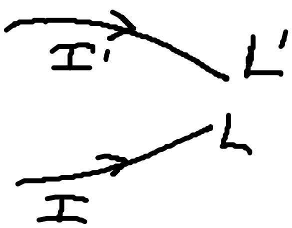
$\underline{B}(\underline{r})$ = magn. Induktion am Ort $\underline{r}$, erzeugt durch andere bewegte Ladungen mit Stromdichte $\underline{j}(\underline{r}')$

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\underline{j}(\underline{r}')\times\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}} \qquad \underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(\underline{r}')\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$$

Ampère-Gesetz

Einheit: (SI): $[B] = 1\,\frac{Ns}{Cm} = 1\,T = 1$ Tesla

$$\Rightarrow \mu_0 = 1{,}26\cdot 10^{-6}\,\frac{Vs}{Am}$$



$$\boxed{\underline{F} = \frac{\mu_0}{4\pi} I I' \int_L d\underline{r} \times \int_{L'} d\underline{r}' \times \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}}$$

Kraft von L' auf L

$$\Longleftrightarrow \quad \boxed{\underline{F} = -\frac{\mu_0}{4\pi} I I' \iint_{LL'} (d\underline{r}\cdot d\underline{r}') \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}}$$

Biot-Savart-Gesetz

parallele Ströme ($I d\underline{r}\, I' d\underline{r}' > 0$): Anziehung

antiparalle Ströme ($I d\underline{r}\, I' d\underline{r}' < 0$): Abstoßung