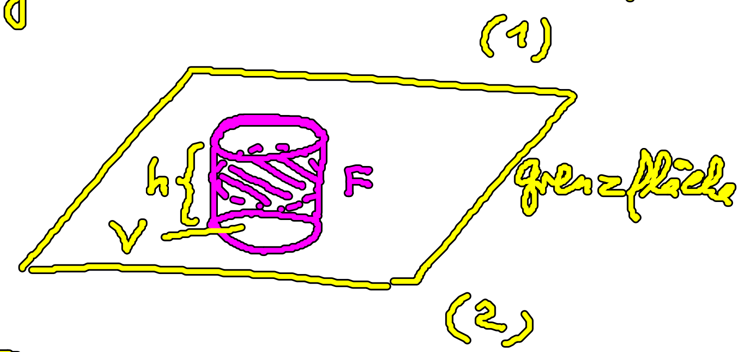
5.4 Grenzbedingungen für Felder

Frage: Wie verhalten sich $\underline{E}$, $\underline{D}$, $\underline{H}$, $\underline{B}$ an Grenzflächen zwischen verschiedenen Materialien?

Integration der Maxwell-Gln. über Volumen V:



$$(1)\quad \int_V d^3r\, \underline{\nabla}\times\underline{E} = -\int_V d^3r\, \underline{\dot{B}}$$

$$(2)\quad \int_V d^3r\, \underline{\nabla}\times\underline{H} = \int_V d^3r\,(\underline{j}+\underline{\dot{D}})$$

$$(3)\quad \int_V d^3r\, \underline{\nabla}\cdot\underline{B} = 0 \overset{\text{Gauß}}{=} \oint_{\partial V} d\underline{f}\cdot\underline{B}$$

$$(4)\quad \int_V d^3r\, \underline{\nabla}\cdot\underline{D} = \int_V d^3r\,\rho \overset{\text{Gauß}}{=} \oint_{\partial V} d\underline{f}\cdot\underline{D}$$

Normalkomponenten

$h \to 0$

$$(3) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f}\cdot\underline{B} = \int_F d\underline{f}\,(\underline{B}^{(1)}-\underline{B}^{(2)})$$

$$= \int_F df\,\underline{n}\,(\underline{B}^{(1)}-\underline{B}^{(2)})$$

$$(4) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f}\cdot\underline{D} = \int_F df\,\underline{n}\,(\underline{D}^{(1)}-\underline{D}^{(2)})$$

Annahme: freie Flächenladungsdichte σ

$$\rho(\underline{r},t) = \sigma(x,y,t)\,\delta(z) \qquad \underline{e}_z = \underline{n}$$

$$\Rightarrow \lim_{h\to 0} \int_V d^3r\, \rho = \int_F df\, \underline{\sigma}$$

bel. Fläche F $\Rightarrow$ $$\boxed{\begin{array}{l} \underline{n}\left(\underline{B}^{(1)} - \underline{B}^{(2)}\right) = 0 \\ \underline{n}\left(\underline{D}^{(1)} - \underline{D}^{(2)}\right) = \sigma \end{array}}$$

Tangentialkomponenten

Verallg. Gauß'scher Satz:

$$(1) \Rightarrow \oint_{\partial V} d\underline{f} \times \underline{E} = -\int_V d^3r\, \dot{\underline{B}}$$

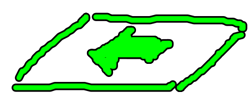
$$(2) \Rightarrow \oint_{\partial V} d\underline{f} \times \underline{H} = \int_V d^3r \left(\underline{j} + \dot{\underline{D}}\right)$$

$\underline{h \to 0}$ $$(1) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f} \times \underline{E} = \int_F df\, \underline{n} \times \left(\underline{E}^{(1)} - \underline{E}^{(2)}\right)$$ Tangentialkomponente

$$(2) \Rightarrow \lim_{h\to 0} \oint_{\partial V} d\underline{f} \times \underline{H} = \int_F df\, \underline{n} \times \left(H^{(1)} - H^{(2)}\right)$$

freie Flächenstromdichte $\underline{g}$:

$$\underline{j}(\underline{r},t) = \underline{g}(x,y,t)\,\delta(z)$$



$$\Rightarrow \lim_{h\to 0} \int_V d^3r\, \underline{j} = \int_F df\, \underline{g}$$

$\underline{E}, \underline{D}$ und $\dot{\underline{B}}, \dot{\underline{D}}$ beschränkt $\Rightarrow$ $$\lim_{h\to 0} \int_V d^3r\, \dot{\underline{B}} = 0$$

$$\lim \int d^3r\, \dot{\underline{D}} = 0$$

für bel. F $\Longrightarrow$

(1) $\int_F df\, \underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)}) = 0$

(2) $\int_F df\, \underline{n} \times (\underline{H}^{(1)} - \underline{H}^{(2)}) = \int_F df\, \underline{g}$

$h \to 0$ ✓

$$\boxed{\begin{aligned} \underline{n} \times (\underline{E}^{(1)} - \underline{E}^{(2)}) &= 0 \\ \underline{n} \times (\underline{H}^{(1)} - \underline{H}^{(2)}) &= \underline{g} \end{aligned}}$$

Zusammenfassung : $\delta \underline{E} := \underline{E}^{(1)} - \underline{E}^{(2)}$

Maxwell-Gln.	Grenzbed.
$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$	$\underline{n} \times \delta\underline{E} = 0$
$\underline{\nabla} \cdot \underline{D} = \rho$	$\underline{n} \cdot \delta\underline{D} = \sigma$
$\underline{\nabla} \times \underline{H} = \underline{j} + \dot{\underline{D}}$	$\underline{n} \times \delta\underline{H} = \underline{g}$
$\underline{\nabla} \cdot \underline{B} = 0$	$\underline{n} \cdot \delta\underline{B} = 0$

Tangentialkomp. v. $\underline{E}$ stetig

Normalkomp. v. $\underline{D}$ springt

Tangentialkomp. v. $\underline{H}$ springt

Normalkomp. v. $\underline{B}$ stetig

5.6 Wellenausbreitung in Materie

$\underline{D} = \epsilon_0 \epsilon\, \underline{E}$ $\quad (\epsilon > 1)$

$\underline{B} = \mu_0 \mu\, \underline{H}$ $\quad$ (i.a. $\mu \simeq 1$)

$\underline{j} = \sigma \underline{E}$ $\quad$ (Ohm'sches Gesetz)

a) Wellen in leitenden Medien ohne Dispersion

(d.h. ϵ, μ, σ unabh. v. ω)

$\rho = 0$

$$\underline{\nabla} \times \underline{E} + \dot{\underline{B}} = 0$$

$$\underline{\nabla} \times \underline{B} - \mu_0\mu\,\epsilon_0\epsilon\,\dot{\underline{E}} = \mu_0\mu\,\underline{j} = \mu_0\mu\,\sigma\,\underline{E} \qquad \Big|\ \underline{\nabla}\cdot\underline{E} = 0,\quad \underline{\nabla}\cdot\underline{B} = 0$$

$$\Rightarrow \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla}(\overbrace{\underline{\nabla}\cdot\underline{E}}^{0}) - \Delta \underline{E}$$

$$= -\underline{\nabla} \times \dot{\underline{B}} = -\mu_0\mu\,\epsilon_0\epsilon\,\ddot{\underline{E}} - \mu_0\mu\,\sigma\,\dot{\underline{E}}$$

$$\boxed{\Delta \underline{E} - \frac{1}{c_M^2}\left(\ddot{\underline{E}} + \frac{\sigma}{\epsilon_0\epsilon}\dot{\underline{E}}\right) = 0}$$

gedämpfte Wellengl. $\quad c_M := \dfrac{1}{\sqrt{\mu_0\mu\,\epsilon_0\epsilon}} = \dfrac{c}{\sqrt{\mu\epsilon}}$

(1-dim. Telegraphengl., beschreibt Drahtwellenausbreitung)

Harmon. ebene Welle

$$\underline{E}(\underline{r},t) = \underline{E}_0\, e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

$$\Rightarrow \boxed{\underline{k}^2 = \frac{\epsilon\mu}{c^2}\,\underline{\omega}^2\left(1 + i\,\frac{1}{\omega\tau}\right)}$$ Dispersionsrelation

$\tau := \dfrac{\epsilon_0\epsilon}{\sigma}$ dielektr. Relaxationszeit

Wellenvektor $k \in \mathbb{C}$ (wegen Dämpfung)

Setze $\boxed{k = \frac{\omega}{c}\tilde{n} = \frac{\omega}{c}(n + i\gamma)}$ c Vakuumlichtgeschw.

$\tilde{n} = n + i\gamma$ komplexer Brechungsindex

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}(n^2 - \gamma^2 + 2in\gamma) \overset{!}{=} \frac{\omega^2}{c^2}\epsilon\mu\left(1 + \frac{i}{\omega\tau}\right)$$

$$\left.\begin{array}{lll} \text{Re} & n^2 - \gamma^2 & = \epsilon\mu \\ \text{Im} & n\gamma & = \frac{\epsilon\mu}{2\omega\tau} \end{array}\right\} \Rightarrow n, \gamma \qquad (*)$$

oBdA $\underline{k} \parallel x_3$: $\underline{E}(x_3, t) = \underline{E}_0 \, e^{-\frac{x_3}{d}} e^{-i\omega\left(t - \frac{n}{c}x_3\right)}$

gedämpfte Welle mit
Phasengeschw. $\frac{c}{n}$
und Extinktionskoeff. $d := \frac{c}{\omega\gamma}$

Isolator $(\sigma = 0)$: $\tau \to \infty \overset{(*)}{\Rightarrow} \gamma = 0$

ungedämpft
reeller Brech.index $\boxed{n = \sqrt{\epsilon\mu} \approx \sqrt{\epsilon} > 1}$

Phasengeschw. $\frac{c}{n} < c$

Metall (σ groß) : $\tau = \frac{\epsilon_0 c}{\sigma} \ll \frac{1}{\omega}$ (für alle Frequ. bis UV)

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}(n^2 - \gamma^2 + 2in\gamma) \approx \frac{\omega^2}{c^2}\epsilon\mu\frac{i}{\omega\tau}$$

$\overset{(*)}{\Longrightarrow}$ $n^2 - \varkappa^2 \approx 0$

$n\varkappa \approx n^2 \approx \varkappa^2 \approx \frac{\epsilon\mu}{2\omega\tau} \Longrightarrow \boxed{n = \varkappa = \sqrt{\frac{\epsilon\mu}{2\omega\tau}}}$

Ext.koeff $d = \frac{c}{\omega\varkappa} \sim$ cm für 100 Hz

$= c\sqrt{\frac{2\tau}{\epsilon\mu\omega}}$ $\sim 10\,\mu m$ für 100 MHz

hochfrequente Wellen dringen nicht in Metall ein!

b) <u>Diel. Dispersion</u> (Ann. $\mu = 1$)

zeitl. Dispersion $\hat{\chi}(\omega)$

$\epsilon = 1 + \chi$

$$\boxed{\hat{\underline{P}}(\omega) = \epsilon_0 \hat{\chi}(\omega) \hat{\underline{E}}(\omega)}$$

$$\hat{\chi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \chi(t)\, e^{i\omega t}$$ (dynam. el. Suszeptib.)

$$\underline{P}(\underline{r}, t) = \frac{\epsilon_0}{\sqrt{2\pi}} \int_{-\infty}^{t} dt'\, \chi(t-t')\, \underline{E}(\underline{r}, t')$$

Gedächtniseffekt?

i.a. $\hat{\chi}(\omega) \in \mathbb{C}$ <u>komplexe diel. Fkt.</u>

$$\epsilon(\omega) = 1 + \hat{\chi}(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$$

<u>monochromat. ebene Welle</u> $\underline{E}(\underline{r}, t) = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

$$\Longrightarrow \boxed{k^2 = \underline{\epsilon(\omega)} \frac{\omega^2}{c^2}\left(1 + i\frac{1}{\omega\tau}\right)} \quad (**)$$

<u>Isolator</u> (dispersives Diel.) $\boxed{k^2 \approx \epsilon(\omega) \frac{\omega^2}{c^2}}$

$$\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$$

$$\tilde{n}(\omega)^2 = \epsilon(\omega) \equiv \epsilon' + i\epsilon''$$

$$(**) \quad \left.\begin{aligned} \epsilon'(\omega) &= n^2 - \kappa^2 \\ \epsilon''(\omega) &= 2n\kappa \end{aligned}\right\} \Rightarrow \boxed{\left.\begin{matrix} \kappa \\ n \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left(\sqrt{\epsilon'^2 + \epsilon''^2} \mp \epsilon\right)^{1/2}}$$

Abs. koeff.: κ

rell. Brech. index n