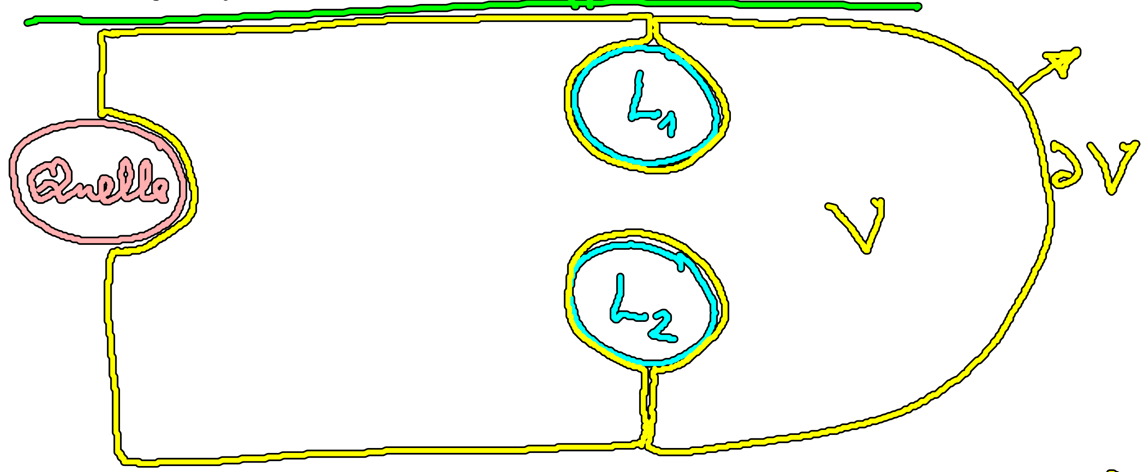
Problem: Randbed. für Φ, $\underline{A}$ sind in Wirklichkeit nicht bekannt, sondern müssen selbstkonsistent bestimmt werden.

Skalare Kirchhoff-Identität



Green'scher Satz: $\displaystyle\oint_{\partial V} d\underline{f}\,(\varphi\underline{\nabla}\psi - \psi\underline{\nabla}\varphi) = \int d^3r\,(\varphi\Delta\psi - \psi\Delta\varphi)$

Setze $\psi(\underline{r}) = \tilde{G}(\underline{r}-\underline{r}')$, $\varphi(\underline{r}) = \Phi(\underline{r})$ sei Lösung

$$\Rightarrow \oint_{\partial V} d\underline{f}\,\left(\Phi\,\underline{\nabla}_r\tilde{G}(\underline{r}-\underline{r}') - \tilde{G}(\underline{r}-\underline{r}')\,\underline{\nabla}_r\Phi\right) = \underbrace{\int_V d^3r\,\Big(\Phi\underbrace{\Delta\tilde{G}}_{-\delta(\underline{r}-\underline{r}')-\cancel{k^2\tilde{G}}} - \tilde{G}\underbrace{\Delta\Phi}_{-\underset{=0}{\frac{\rho}{\varepsilon_0}}-\cancel{k^2\Phi}}\Big)}_{-\Phi(\underline{r}')}$$

$$\Rightarrow \boxed{\Phi(\underline{r}') = \oint_{\partial V} d\underline{f}\,\left[\tilde{G}(\underline{r}-\underline{r}')\,\underline{\nabla}_r\Phi(\underline{r}) - \Phi(\underline{r})\,\underline{\nabla}_r\tilde{G}(\underline{r}-\underline{r}')\right]}$$

Pot. $\Phi(\underline{r}')$ im Innern von V festgelegt! $\underline{r}' \in V$

(a) Green'sche Fkt. des unendl. Raumes:

Randbed. $\tilde{G}(\underline{r}-\underline{r}') \to 0$ für $r \to \infty$

$\Longrightarrow$ retardiertes Pot. (§4.2)

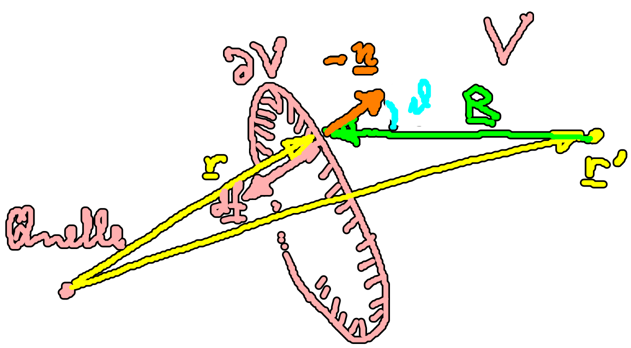
$$G(\underline{r}-\underline{r}', \underbrace{t-t'}_{\tau}) = \begin{cases} \dfrac{1}{4\pi|\underline{r}-\underline{r}'|}\,\delta\!\left(\tau - \dfrac{|\underline{r}-\underline{r}'|}{c}\right) & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

$$\boxed{\tilde{G}(\underline{r}-\underline{r}') := \int_0^\infty d\tau\, G(\underline{r}-\underline{r}', \tau)\, e^{i\omega\tau} = \frac{e^{ik|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|}} \quad \text{mit } k := \frac{\omega}{c}$$

$$\phi(\underline{r},t) = \int_{\mathbb{R}^3} d^3r'\, \tilde{G}(\underline{r}-\underline{r}')\, e^{-i\omega t}\, \rho(\underline{r}')/\epsilon_0$$

$$= \int d^3r'\, \frac{e^{i(k|\underline{r}-\underline{r}'| - \omega t)}}{4\pi|\underline{r}-\underline{r}'|}\, \rho(\underline{r}')/\epsilon_0$$

beschreibt Überlagerung auslaufender Kugelwellen
(Ausstrahlbed.)



$\underline{R} := \underline{r} - \underline{r}'$

Kirchhoff-Identität:

$$\phi(\underline{r}') = \frac{1}{4\pi}\int_{\partial V} d\underline{f}_R \left\{ \frac{e^{ikR}}{R}\, \underline{\nabla}_r \phi(\underline{r}) - \phi(\underline{r})\, \underbrace{\underline{\nabla}_r \frac{e^{ikR}}{R}}_{\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|}} \right\}$$

mit $d\underline{f}_R \cdot \dfrac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|} = df_R \cos\vartheta$

und Beschränkung auf Fernzone von ∂V (d.h. $R \gg \frac{1}{k}$):

$$\boxed{\phi(\underline{r}') \approx \frac{1}{4\pi}\int_{\partial V} df_R\, \underbrace{\left\{ \frac{\partial}{\partial n}\phi(\underline{r}) - ik\,\phi(\underline{r})\cos\vartheta \right\}}_{\text{richtungsabh. Amplitude}}\, \underbrace{\frac{e^{ikR}}{R}}_{\text{Kugelwelle}}}$$

„Sekundärwelle“

Exakte Formulierung des Huygens'schen Prinzips

(Jeder Pkt. der Oberfläche eines Hindernisses ist Ausgangspkt. einer Kugelwelle, deren phasengerechte Überlagerung ergibt das Wellenfeld in $\underline{r}'$.)

(b) Greenfkt. zu Randbed. $\tilde{G}(\underline{r}-\underline{r}')\Big|_{\substack{\underline{r}\in\partial V\\ \underline{r}'\in V}}=0$

$$\Rightarrow \boxed{\phi(\underline{r}') = -\int_{\partial V} d\underline{f}\, \phi(\underline{r})\,\underline{\nabla}_r \tilde{G}(\underline{r}-\underline{r}')}$$

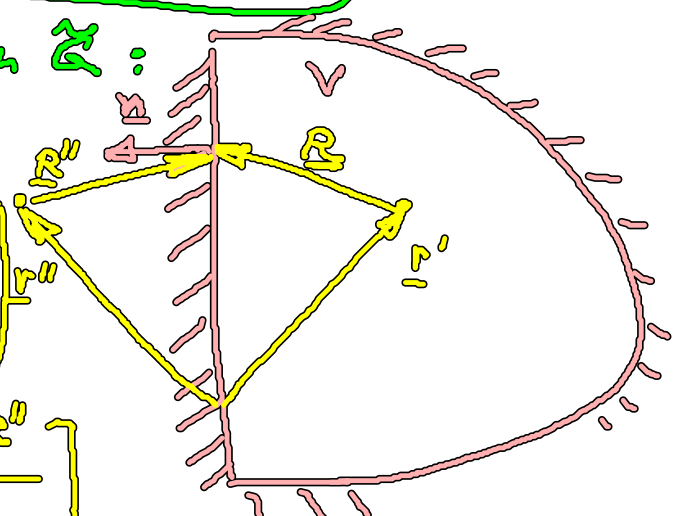
neue Greenfkt. $\tilde{G}(\underline{R}) = g(\underline{R}) + \underbrace{\frac{1}{4\pi}\frac{e^{ikR}}{R}}_{\text{alte Greenfkt.}}$

$\longrightarrow 0$ für $r\to\infty$

$$\boxed{(\Delta+k^2)g=0 \quad \text{mit } g\Big|_{\partial V} = -\frac{1}{4\pi}\frac{e^{ikR}}{R}\Big|_{\partial V}}$$

Beispiel für die Konstr. von $\tilde{G}$:

Ebener Schirm



$$\tilde{G}(\underline{r}-\underline{r}') = \frac{1}{4\pi}\left(\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} - \frac{e^{ik|\underline{r}-\underline{r}''|}}{|\underline{r}-\underline{r}''|}\right)\Big|_{\underline{r}''}$$

$$\underline{\nabla}_r \tilde{G}(\underline{r}-\underline{r}') = \frac{1}{4\pi}\left[\underline{\nabla}_r \frac{e^{ikR}}{R} - \underline{\nabla}_r \frac{e^{ikR''}}{R''}\right]$$

$$= \frac{1}{4\pi}\left[\frac{e^{ikR}}{R}\left(ik-\frac{1}{R}\right)\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|} - \frac{e^{ikR''}}{R''}\left(ik-\frac{1}{R''}\right)\frac{\underline{r}-\underline{r}''}{|\underline{r}-\underline{r}''|}\right]$$

mit $R = R''$, $\quad d\underline{f}\cdot\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|} = -d\underline{f}\,\frac{\underline{r}-\underline{r}''}{|\underline{r}-\underline{r}''|} = df\cos\alpha$

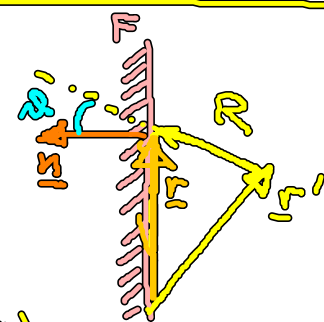
folgt $d\underline{f}\cdot\nabla_r \tilde{G} = df\,\frac{1}{2\pi}\,\frac{e^{ikR}}{R}\left(ik - \cancel{\frac{1}{R}}\right)\cos\vartheta$

Fernzone $\lambda \ll R$

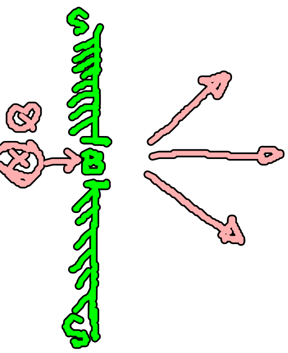
Fernzone:

$$\boxed{\Phi(\underline{r}') = -\int_{\partial V}\left[d\underline{f}\cdot\nabla_r\tilde{G}(\underline{r}-\underline{r}')\right]\Phi(\underline{r}) = -\frac{i}{\lambda}\int_F df\,\Phi(\underline{r})\,\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\cos\vartheta}$$

Randwerte $\Phi(\underline{r})\big|_{r\in F}$ erraten!



Kirchhoff'sche Näherung:



Annahme: $\Phi(\underline{r})\big|_S = 0$ (Leiter)

$$\Phi(\underline{r})\big|_B = \frac{e^{ikR^Q}}{R^Q}$$ freie einfallende Welle

$$\Rightarrow \Phi(\underline{r}') = -\frac{i}{\lambda}\int_B df\,\frac{e^{ik(R+R^Q)}}{R\,R^Q}\cos\vartheta$$

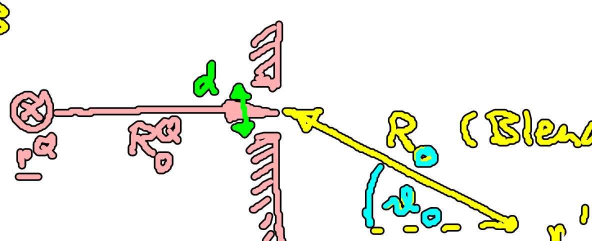
$\underline{R} = \underline{r} - \underline{r}'$

$\underline{R}^Q = \underline{r} - \underline{r}^Q$

$df = d^2r$

Kleine Blende

(aber $\lambda \ll d$)



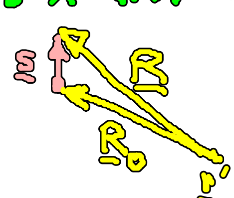
(Blendenschwerpkt.)

$\cos\vartheta = \cos\vartheta_0 \approx$ const.

$$\boxed{\Phi(\underline{r}') \approx -\frac{i}{\lambda}\,\frac{\cos\vartheta_0}{R_0 R_0^Q}\int_B df\, e^{ik(R+R^Q)}}$$

Grenzfälle:

(i) Fraunhofer'sche Beugung (Fernzone: $\lambda \ll d \ll R$)



Setze $\underline{R} = \underline{R}_0 + \underline{s}$

$R^2 \approx R_0^2 + 2\underline{R}_0\cdot\underline{s}$

$R \approx R_0 + \underline{\alpha}\cdot\underline{s}$ $\qquad \underline{\alpha} := \frac{\underline{R}_0}{R_0}$

analog $R^Q \approx R_0^Q + \underline{\alpha}_0 \cdot \underline{s}$ $\quad \underline{\alpha}_0 := \frac{\underline{R}_0^Q}{R_0^Q}$

$$\Rightarrow \boxed{\Phi(\underline{r}') \approx -\frac{i}{\lambda} \frac{e^{ik(R_0+R_0^Q)}}{R_0 R_0^Q} \cos\vartheta_0 \int_B d^2s \; e^{ik(\underline{\alpha}+\underline{\alpha}_0)\cdot\underline{s}}}$$

(ii) Fresnel'sche Beugung (Mittelzone $\lambda \ll R \approx d$)

$$R^2 = R_0^2 + 2\underline{R}_0 \underline{s} + s^2$$ nicht genähert!