2.2 Fokker-Planck-Gleichung

Zeitentwicklung eines kontinuierlichen Markov-Prozesses:

$X(t)$ (1-dim. Zufallsvar.)

$$\boxed{\frac{\partial}{\partial t} p(x,t|x_0,t_0) = \underbrace{-\frac{\partial}{\partial x}\left[A(x,t)p(x,t|x_0,t_0)\right]}_{\text{Drift}} + \underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)p(x,t|x_0,t_0)\right]}_{\text{Diff.}}}$$

Anfangsbed. $p(x,t_0|x_0,t_0) = \delta(x-x_0)$

$\hat{=}$ Ein-Zeit-Wahrsch. $p(x,t) = \int dx_0\, p(x,t;x_0,t_0) = \int dx_0\, p(x,t|x_0,t_0)p(x_0,t_0)$
mit Anf.bed. $p(x,t_0)$ (weniger singulär)

Randbed. (n-dim.)

Fokker-Planck (FP)-Gl. ist lokale Bilanzgl.

$$\frac{\partial p(\underline{x},t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i} J_i(\underline{x},t) \qquad (\dot{p} + \operatorname{div} \underline{J} = 0)$$

(vgl. Teilchendichte ρ
$\dot{\rho} + \operatorname{div}\underline{J} = 0$
$\underline{J} = \underline{v}\rho - D\underline{\nabla}\rho$
$\Rightarrow \dot{\rho} = -\rho\underline{\nabla}\underline{v} + D\Delta\rho$)

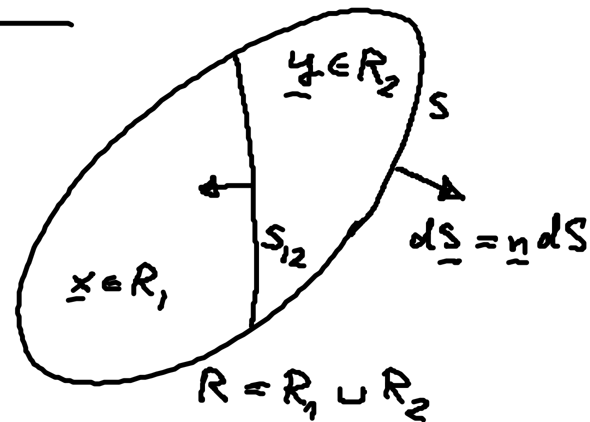
mit Wahrscheinl.strom

$$J_i(\underline{x},t) = A_i(\underline{x},t)p(\underline{x},t) - \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}\left(B_{ij}(\underline{x},t)\,p(\underline{x},t)\right)$$

Globale Bilanzgl. für Gebiet $R \in \mathbb{R}^n$

$$P(R,t) := \int_R d^n x\, p(\underline{x},t)$$

$$\frac{\partial P}{\partial t} = -\int_S d\underline{S}\cdot\underline{J}(\underline{x},t) \quad \left(\begin{array}{c}\text{gauß'scher}\\ \text{Satz}\end{array}\right)$$



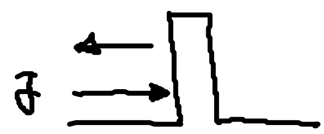
Netto-Wahrsch.fluss durch beliebige Fläche S_{12}:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{R_1} d^n x \int_{R_2} d^n y \left[p(\underline{x}, t+\Delta t; \underline{y}, t) - p(\underline{y}, t+\Delta t; \underline{x}, t) \right] = \int_{S_{12}} d\underline{S} \cdot \underline{J}(\underline{x}, t)$$

$$R_1 \leftarrow R_2$$

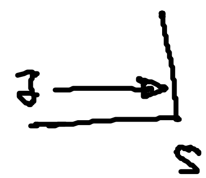
(a) <u>Reflektierende Barriere</u>:

$$\underline{n} \cdot \underline{J}(\underline{x}, t)\Big|_{\underline{x} \in S} = 0$$

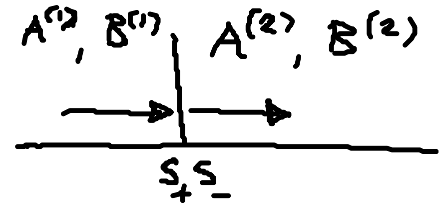


(b) <u>Absorbierende Barriere</u>:

$$p(\underline{x}, t)\Big|_{\underline{x} \in S} = 0$$



(c) <u>Grenzfläche</u> zwischen 2 Medien:

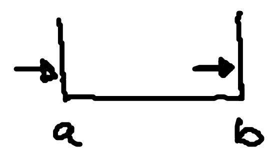


$$\underline{n} \cdot \underline{J}\Big|_{S_+} = \underline{n} \cdot \underline{J}\Big|_{S_-}$$

$$p\Big|_{S_+} = p\Big|_{S_-}$$

(d) <u>Periodische Randbed.</u>:

$$p(a,t) = p(b,t)$$
$$J(a,t) = J(b,t)$$



(e) <u>Natürliche Randbed.</u>: $A(a,t) = 0$

(Geschw. $= 0$, Rand wird nie erreicht)

<u>Stationäre Lösung für homogenen Markov-Prozess</u>

homogen $\Rightarrow$ A, B unabh. von t

1-dim.: $\frac{d}{dx} J(x,t) = 0 \quad \Rightarrow J(x) = \text{const.} = J(a) = J(b)$

(i) <u>reflekt. Randbed.</u> $\Rightarrow J(x) = 0$

$$\Rightarrow \quad A(x)\,p^*(x) = \frac{1}{2}\frac{d}{dx}\left[B(x)\,p^*(x)\right] = 0 \qquad \Rightarrow 2\frac{A}{B}\,dx = \frac{d(Bp^*)}{Bp^*}$$

$$\Rightarrow \boxed{p^*(x) = \frac{N}{B(x)} \exp\left[2\int_a^x dx' \,\frac{A(x')}{B(x')}\right]}$$

Potenziallösung, Normierung $\int_a^b dx\, p^*(x) = 1 \Rightarrow N$

(ii) <u>period. Randbed.</u> $\Rightarrow J(x) = J$

$$A(x)\,p^*(x) - \frac{1}{2}\frac{d}{dx}\left[B(x)\,p^*(x)\right] = J \qquad (1) \quad \text{lin. inhom. Dgl.}$$

Mit $\psi(x) := \exp\left[2\int_a^x dx' \frac{A(x')}{B(x')}\right]$ ergibt sich

$$A\,\frac{\psi}{B} - \frac{1}{2}\frac{d}{dx}\psi = 0 \qquad (\text{homog. Lös } p^* = \frac{\psi}{B})$$

$$\Rightarrow A = \frac{B}{2}\frac{\psi'}{\psi} \quad \overset{\text{in }(1)}{\Longrightarrow} \quad Bp^*\frac{\psi'}{\psi} - (Bp^*)' = 2J$$

$$\Rightarrow \frac{-(Bp^*)\psi' + (Bp^*)'\psi}{\psi^2} = -\frac{2J}{\psi}$$

$$\Rightarrow \left(\frac{Bp^*}{\psi}\right)' = -\frac{2J}{\psi}$$

$$\overset{\int_a^x dx'}{=} \quad \left.\frac{Bp^*}{\psi}\right|_a^x = -2J\int_a^x \frac{dx'}{\psi(x')} \quad \Rightarrow p^*(x)$$

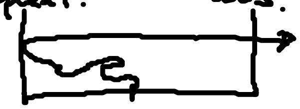
Bestimmung von J durch period. Randbed.

$$\Rightarrow p^*(x) = p^*(a)\,\frac{\int_a^x \frac{dx'}{\psi(x')}\frac{B(b)}{\psi(b)} + \int_x^b \frac{dx'}{\psi(x')}\frac{B(a)}{\psi(a)}}{\frac{B(x)}{\psi(x)}\int_a^b \frac{dx'}{\psi(x')}}$$

<u>First passage time</u>

Fragestellung: Wie lange hält sich ein Teilchen in einem vorgegebenen Gebiet auf?

<u>Teilchen zwischen 1 absorb. u. 1 reflektierenden Barriere</u>



Ziel: Entweichzeit T (escape time)

Wahrsch., dass das Teilchen zur Zeit t noch in (a,b), wenn es bei x gestartet ist:

$$G(x,t) := \int_a^b dx'\, p(x',t\,|\,x,0) \equiv \mathrm{Prob}\,(T \geq t)$$

stationärer Prozess: $p(x',t\,|\,x,0) = p(x',0\,|\,x,-t)$

Rückwärts-FP-gl.: Rückwärtsentw. für $t'<t$ aus (x,t)

$$\frac{\partial p(x,t\,|\,y,t')}{\partial t'} = -A(y,t')\frac{\partial p(x,t\,|\,y,t')}{\partial y} - \frac{1}{2}B(y,t')\frac{\partial^2 p(x,t\,|\,y,t')}{\partial y^2}$$

homog. (A,B zeitunabh.): $\displaystyle \frac{\partial}{\partial t}G(x,t) = -\frac{\partial}{\partial t'}\int_a^b dx'\, p(x',0\,|\,x,\underset{-t}{\underset{\shortparallel}{t'}})$

$$\frac{\partial}{\partial t}G(x,t) = A(x)\frac{\partial}{\partial x}G(x,t) + \frac{1}{2}B(x)\frac{\partial^2}{\partial x^2}G(x,t)$$

„End" bed.: $p(x',0\,|\,x,0) = \delta(x-x')$

$$\Rightarrow \quad G(x,0) = \begin{cases} 1 & a \leq x \leq b \\ 0 & \text{sonst} \end{cases}$$

absorb. Randbed.: $\mathrm{Prob}\,(T \geq t) = 0$ wenn $x=a$ oder b

$$\Leftrightarrow \quad G(a,t) = G(b,t) = 0$$

<u>mittlere erste Übergangszeit</u> (mean first passage time)

$$T(x) := \langle T \rangle = -\int_0^\infty t\, dG = -\int_0^\infty dt\, t \frac{\partial}{\partial t}G(x,t) \overset{\text{part. Int.}}{=} \int_0^\infty dt\, G(x,t)$$

Dgl. für $T(x)$ aus der Rückwärts-FP-Gl. ..

$$\int_0^\infty \frac{\partial}{\partial t} G(x,t)\,dt = \underbrace{G(x,\infty)}_{0} - \underbrace{G(x,0)}_{1} = -1$$

$$** \qquad A(x)\frac{\partial}{\partial x}T(x) + \frac{1}{2}B(x)\frac{\partial^2}{\partial x^2}T(x) = -1 \quad , \quad \text{Randbed. } T(a)=T(b)=0$$

Lösung der hom. Gl. $A\psi - \frac{1}{2}B\frac{\partial}{\partial x}\psi = 0 \Leftrightarrow 2\frac{A}{B}dx = \frac{d\psi}{\psi}$

$$\Leftrightarrow \boxed{\psi(x) = \exp\left[2\int_a^x dx' \,\frac{A(x')}{B(x')}\right]}$$

$\Rightarrow$ Lösung der <u>inhom.</u> Gl. $T(x)$ durch $\psi(x)$ ausgedrückt:

$$\boxed{T(x) = 2\int_x^b \frac{dy}{\psi(y)} \int_a^y dz \,\frac{\psi(z)}{B(z)}} \qquad (*)$$

<u>Beweis</u>: $T' = -\frac{2}{\psi(x)} \int_a^x dz \,\frac{\psi(z)}{B(z)}$

$$T'' = -\frac{2}{\psi(x)^2}\left[\frac{\psi(x)^2}{B(x)} - \underset{\sim}{\psi'(x)}\int_a^x dz \,\frac{\psi(z)}{B(z)}\right]$$

$$\underset{\sim}{\psi'} = \frac{2A}{B}\psi \Rightarrow \frac{2A}{\psi}$$

(**)

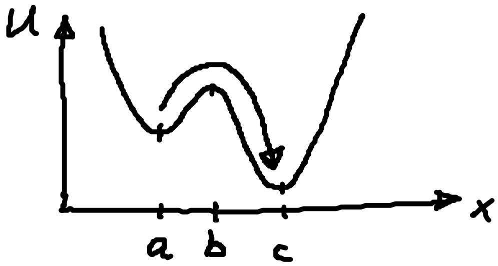
$$\Rightarrow AT' + \frac{1}{2}BT'' = \underbrace{-\frac{2A}{\psi}\int dz \,\frac{\psi}{B}} - \underbrace{\left[1 - \frac{B}{\psi^2}\underset{\sim}{\psi'}\int dz \,\frac{\psi}{B}\right]} \overset{!}{=} -1$$

Randbed.: $T'(a) = \int_0^\infty dt \,\frac{\partial}{\partial x} G(x,t)\Big|_{x=a} = 0$ ✓

$T(b) = 0$ ✓

□

Anwendung: Kramers' Problem (1940)



Entweichen über Potenzialbarriere:

bistab. Pot. $U(x)$

überdämpftes Teilchen: $\dot{x} = -U'(x) \equiv \underset{\text{Kraft}}{A(x)}$

Diff. konst. $D = \frac{B}{2}$

FP-gl. $\boxed{\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\left[U'(x)\, p(x,t)\right] + D\frac{\partial^2}{\partial x^2} p(x,t)}$

stat. lös. $p^*(x) = \mathcal{N} \exp\left[-\int_a^x dx' \frac{U'(x')}{D}\right] = \tilde{\mathcal{N}} \exp\left[-\frac{U(x)}{D}\right]$

(reflekt. Randbed.)