12.5 Lagrangesche Gleichungen 2. Art

• generalisierte Koord.: $q_1, \dots, q_f$, $f = 3N - Z$

$$\longrightarrow \quad \underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t) \qquad (12.4)$$

• Ziel: Beschreibe Dynamik der $q_1, \dots, q_f \xrightarrow{(12.4)} \underline{r}_i(t)$

kein Zugang zu Zwangskräften

• Hilfsformeln: (i) $\dot{\underline{r}}_i \underset{(12.4)}{=} \sum_j \frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \underline{r}_i}{\partial t}$ (12.22)

(ii) $\delta \underline{r}_i = \sum_j \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j$, $\delta t = 0$ (12.23)

• d'Alembert: $\sum_i (\underline{F}_i^{(t)} - m_i \ddot{\underline{r}}_i) \cdot \delta \underline{r}_i = 0$ (12.15)

(i) $\sum_i \underline{F}_i^{(t)} \cdot \delta \underline{r}_i \overset{(12.23)}{=} \sum_j \underbrace{\sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j}}_{Q_j} \delta q_j = \sum_j Q_j \delta q_j$

$$\boxed{Q_j = \sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j}} \qquad (12.25)$$

... generalisierte Kraft

(ii) $\sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i \overset{(12.23)}{=} \sum_j \underbrace{\sum_i m_i \ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}} \delta q_j$

$\frac{d}{dt}(\dots) = \frac{\partial}{\partial t} \dots + \sum_j \frac{\partial \dots}{\partial q_j} \dot{q}_j$

$$\frac{d}{dt}\Big(m_i \dot{\underline{r}}_i \cdot \underbrace{\frac{\partial \underline{r}_i}{\partial q_j}}_{\overset{(12.22)}{=} \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}}\Big) - m_i \dot{\underline{r}}_i \cdot \underbrace{\frac{d}{dt}\Big(\frac{\partial \underline{r}_i}{\partial q_j}\Big)}_{= \frac{\partial}{\partial q_j} \dot{\underline{r}}_i}$$

$$= \sum_j \sum_i \left\{ \underbrace{\frac{d}{dt}\left(m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}\right)}_{= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m \dot{\underline{r}}_i^2\right)} - \underbrace{m_i \dot{\underline{r}}_i \cdot \frac{\partial}{\partial q_j} \dot{\underline{r}}_i}_{\frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right)} \right\} \delta q_j$$

mit $\boxed{T = \sum_i \frac{1}{2} m_i \dot{\underline{r}}_i^2}$

... kinetische Energie

$$\longrightarrow \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_j \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} T - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

(12.15) (i) − (ii) = 0

$$\longrightarrow \sum_j \left\{ Q_j - \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \right\} \delta q_j = 0$$

alle unabhängig voneinander:
holonome Bindung!
$\delta q_k \neq 0, \quad \delta q_j = 0, \quad j \neq k$

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \qquad (12.26)$$

... Lagrangesche Gln. 2. Art (im allgemeinen)

Sonderfälle

a) konservative Systeme: $\boxed{\text{wichtig!!}}$

$$\boxed{\underline{F}_i^{(t)} = -\underline{\nabla}_i U} \; (12.27) \quad \text{mit } U = U(\underline{r}_1, \ldots, \underline{r}_N)$$

$$= U(\underline{r}_1(\ldots, q_j, \ldots, t) \ldots \underline{r}_N(\ldots))$$

damit: $Q_j \overset{(12.25)}{=} \sum_i \underline{F}_i^{(t)} \cdot \frac{\partial \underline{r}_i}{\partial q_j} = -\sum_i \underline{\nabla}_i U \cdot \frac{\partial \underline{r}_i}{\partial q_j}$

$$\longrightarrow \boxed{Q_j = -\frac{\partial U}{\partial q_j}} \qquad (12.28)$$

also: (12.26) $\longrightarrow \frac{d}{dt} \frac{\partial (T-U)}{\partial \dot{q}_j} - \frac{\partial (T-U)}{\partial q_j} = 0$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = 0} \quad (12.29)$$

... Lagrangesche Gln.

(" " 2. Art, im speziellen)

mit $\boxed{L = T - U}$ (12.30)

... Lagrange Funktion

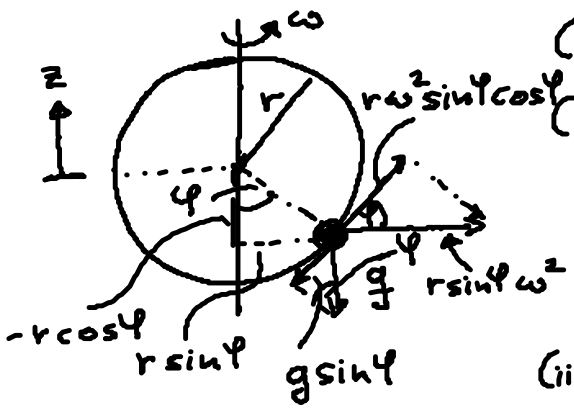
- Bsp 1. Teilchen in 3D. $\underline{r} = (x_1, x_2, x_3) \longrightarrow q_j = x_j \;, \; j = 1,2,3$

$$\left.\begin{aligned} T &= \frac{1}{2} m \sum_j \dot x_j^2 \\ U &= U(x_1, x_2, x_3) \end{aligned}\right\} L = T - U$$

$$\frac{\partial L}{\partial x_j} = -\frac{\partial U}{\partial x_j} = F_j \;, \quad \frac{\partial L}{\partial \dot x_j} = m \dot x_j$$

$$\xRightarrow{(12.29)} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot x_j} - \frac{\partial L}{\partial x_j} = 0 \longrightarrow m \ddot x_j = F_j \quad ✓$$

- Bsp 2: Perle im rotierenden Drahtring (ohne Reibung)



(i) generalisierte Koord.: φ

(ii) Geschw. komp: $\underbrace{r\dot\varphi}_{\text{auf Ring}}$ & $\underbrace{\omega r \sin\varphi}_{\text{rot. Ring}}$

$$\longrightarrow T = \frac{m}{2} r^2 (\dot\varphi^2 + \omega^2 \sin^2\varphi)$$

(iii) $U = mgz = -mg\underline{r\cos\varphi}$

$$\begin{aligned} \longrightarrow L &= T - U \\ &= \frac{m}{2} r^2 (\dot\varphi^2 + \omega^2 \sin^2\varphi) + mgr\cos\varphi \end{aligned}$$

(iv) Bewgl.:
$$\begin{aligned} \frac{\partial L}{\partial \varphi} &= mr^2\omega^2 \sin\varphi\cos\varphi - mgr\sin\varphi \\ &= mr(r\omega^2 \sin\varphi\cos\varphi - g\sin\varphi) \end{aligned}$$

$$\frac{\partial L}{\partial \dot\varphi} = mr^2 \dot\varphi$$

$$\longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = mr\left[r\ddot\varphi - \left(r\omega^2 \sin\varphi\cos\varphi - g\sin\varphi\right)\right] = 0$$

Beschl. unm — Komp. Zentrifugalbeschl. — Erdbeschl. ∥ Ring

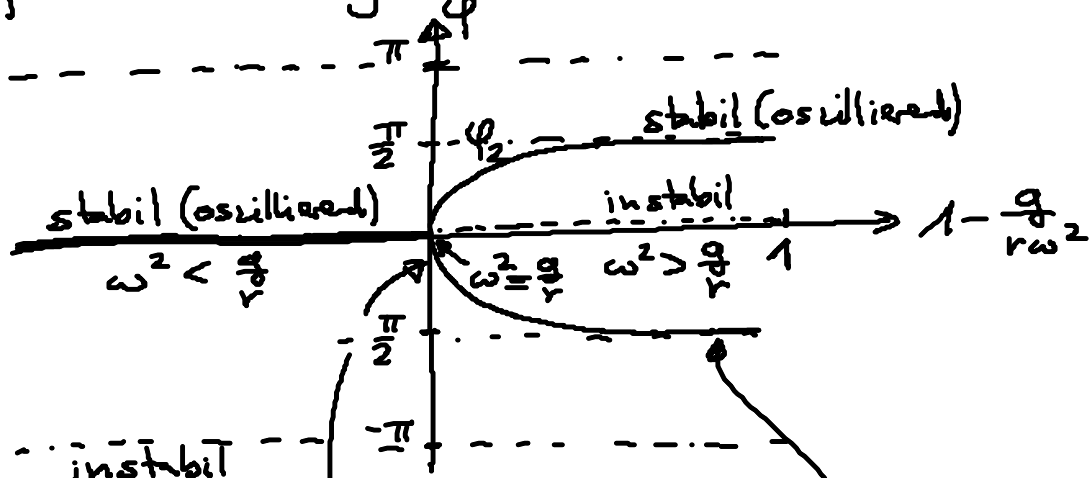
(v) Diskussion:

1. Stationäre Lsg: $\dot\varphi = 0 \longrightarrow \ddot\varphi = 0 \longrightarrow \sin\varphi_1 = 0 \longrightarrow \varphi_1 = 0^\circ, 180^\circ$

$\cos\varphi_2 = \frac{g}{r\omega^2}$

2. Verhalten gegen Störungen: [s. Übungen]

3. Bifurkationsdiagramm: für stationäre Lsg.



Zentrifugalbeschl. treibt die Kugel aus $\varphi = 0$ heraus!

$$\cos\varphi_2 \approx 1 - \frac{\varphi_2^2}{2} = \frac{g}{r\omega^2}$$

$$\longrightarrow \varphi_2 = \pm\sqrt{2}\sqrt{1-\frac{g}{r\omega^2}} \quad !$$

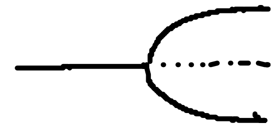
$\hat=$ Bifurkation der $\varphi = 0$ Lsg in zwei gleichwertige Lsg. (Symmetriebrechung!)

NB: <u>Bifurkation</u> = qualitative Änderung der Lsg. eines Systems von Dgl. bei Durchstimmen eines Parameters!

$\longrightarrow$ dynamische Systeme

charakt. Bifurkationstypen:

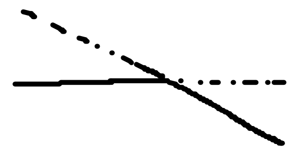
1D: (i) Pitchfork (Mist-/Heu-gabel)-Bifurkation



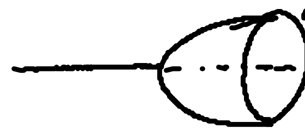
(ii) Saddle-node (Sattel-Knoten)-Bifurkation

„Annihilation" von stabile und instabile Fixpkt.

(iii) Transcritical (transkritische) Bifurkation



2D: z.B. Hopf-Bifurkation



von „Normalformen"