13.2 Eulersche Variationsrechnung

- 1D-Fall: Variationsproblem:

  Geg: Fkt. $y(x)$

  Funktional $F[y(x)] = \int_{x_1}^{x_2} f(y, \underset{\frac{dy}{dx}}{y'}, x)\, dx$ (13.4)

  ... ordnet $y(x)$ eine Zahl zu!

  Ges: $y(x)$, so daß $F[y(x)]$ extremal für feste $y(x_1)$, $y(x_2)$

- diskretisierte Version:

  (i) zur Verdeutlichung

  (ii) Vorgehen in der Numerik

$$\xrightarrow[x_1 = x^{(1)}\; x^{(2)}\; x^{(3)}\; \cdots\cdots\; x_2 = x^{(n)}]{\Delta x} x$$

... Stützstellen

$$y(x^{(i)}) \longrightarrow y_i$$

$$y'(x^{(i)}) \longrightarrow \frac{y_{i+1} - y_i}{\Delta x}$$

$$f(y, y', x) \longrightarrow f\left(y_i, \frac{y_{i+1} - y_i}{\Delta x}, x^{(i)}\right)$$

$$\int \ldots dx \longrightarrow \sum_i \ldots \Delta x$$

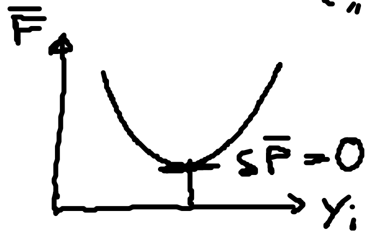
$$F[y(x)] \longrightarrow \bar{F}(y_1, \ldots, y_n)$$

Variation von $\bar{F}$: $\delta \bar{F} = \bar{F}(y_1 + \delta y_1, \ldots, y_n + \delta y_n) - \bar{F}(y_1, \ldots, y_n)$

$$\overset{\text{Taylor}}{\approx} \sum_{i=2}^{n-1} \frac{\partial \bar{F}}{\partial y_i} \delta y_i \quad (13.5)$$

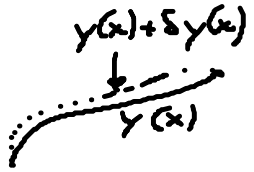
Extremum von $\bar{F}$: $\delta \bar{F} = 0$, beliebiges $\delta y_i \Longleftrightarrow \frac{\partial \bar{F}}{\partial y_i} = 0$ (13.6)

(„horizontale Tangente")



- kontinuierliche Version:

Variation von F:

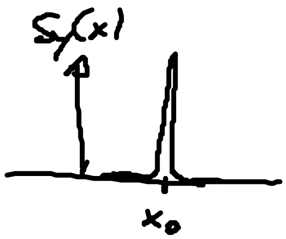


$$\boxed{\begin{aligned} \delta F &= F[y(x)+\delta y(x)] - F[y(x)] \\ &\overset{!}{=} \int \underbrace{\frac{\delta F}{\delta y(x)}}_{\text{„Funktional-ableitung"}} \delta y(x)\, dx \end{aligned}} \quad (13.7)$$

Vgl. mit (13.5): (i) $\sum_i \longrightarrow \int \dots dx$

(ii) $\frac{\partial \bar{F}}{\partial y_i} \longrightarrow \frac{\delta F}{\delta y(x)}$!

Extremum von F:



$$\boxed{\begin{aligned} &\delta F = 0, \quad \text{beliebiges } \delta y(x) \\ &\longrightarrow \frac{\delta F}{\delta y(x)} = 0 \end{aligned}} \quad (13.8)$$

$\longrightarrow$ Ges.: $\frac{\delta F}{\delta y(x)}$!

- Durchführung der Variation für (13.4)

$$\delta F = \delta \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

$$= \int_{x_1}^{x_2} \delta f(y(x), y'(x), x)\, dx$$

$$= \int_{x_1}^{x_2} \Big[\frac{\partial f}{\partial y} \delta y + \underline{\frac{\partial f}{\partial y'} \delta y'}\Big] dx \quad (13.9) \qquad [\delta x = 0!]$$

wichtige Schritte:

(i) $$\delta y' = \delta\left[\lim_{\varepsilon\to 0}\frac{y(x+\varepsilon)-y(x)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0}\frac{\delta y(x+\varepsilon)-\delta y(x)}{\varepsilon}$$

$$\longrightarrow \boxed{\delta y' = \frac{d}{dx}\,\delta y(x)} \qquad (13.10)$$

(ii) in (13.9): mit (13.10) $$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\frac{d}{dx}\,\delta y(x)\,dx$$

$$\overset{\text{part. Integ}}{=} \int_{x_1}^{x_2}\Big\{\underbrace{\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\,\delta y(x)\right]}_{\frac{\partial f}{\partial y'}\,\delta y(x)\Big|_{x_1}^{x_2}=0} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y(x)\Big\}\,dx \quad !!! \qquad (13.11)$$

weil $\delta y(x_1) = \delta y(x_2) = 0$

$$\longrightarrow \delta F \overset{(13.11)}{=} \int_{x_1}^{x_2}\underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]}_{\frac{\delta F}{\delta y(x)}\ !!\ \text{in (13.7)}}\delta y(x)\,dx \qquad (13.12)$$

$$\xrightarrow[\text{(13.7)}]{\text{vgl. mit}} \boxed{\frac{\delta F}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}} \qquad (13.13)$$

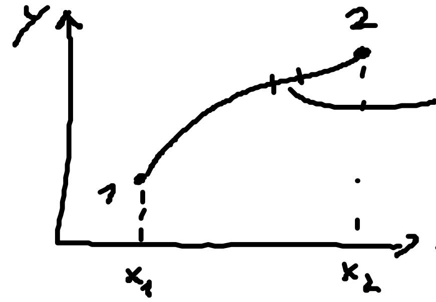
$$\boxed{\delta F = 0 \longleftrightarrow \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0} \qquad (13.14)$$

... Euler-Lagrange-Gl. des Variationsproblems

- **Bsp:**

1. $\boxed{\text{Geodäten = Kurven mit kürzesten Abstand zwischen 2 Pktn}}$

a) <u>Ebene</u>:



Wegelement: $ds = \sqrt{dx^2 + dy^2}$
$= dx \sqrt{1 + \underbrace{\left(\frac{dy}{dx}\right)^2}_{=(y')^2}}$ (13.5)

– Minimiere: $L = \int_1^2 ds = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y')^2}}_{\ell}\, dx$

$$\delta L = 0 \xrightarrow[(13.14)]{} \frac{d}{dx}\frac{\partial \ell}{\partial y'} - \underbrace{\frac{\partial \ell}{\partial y}}_{=0} = 0$$

$$\longrightarrow \frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0 \longrightarrow \frac{y'}{\sqrt{1+(y')^2}} = \text{const}$$

$$\longrightarrow y' = c_1 \longrightarrow \boxed{y = c_1 x + c_2}$$

... Gerade ✓

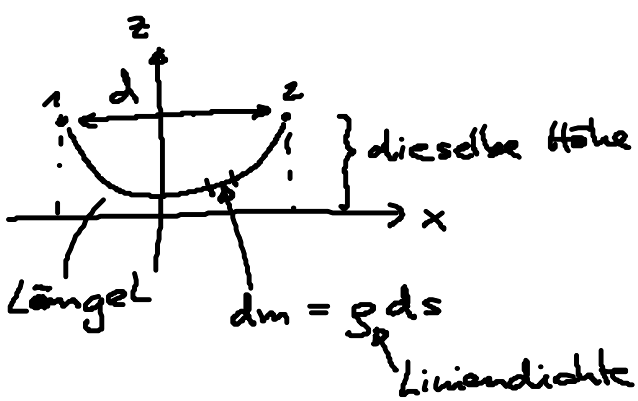
b) Kugel: ⟶ Großkreise

Anwendg.: Routen von Flugzeugen

c) ART: Geodäten = Bahnkurven von Massepktn in gekrümmter 4D Raumzeit

2. Kettenlinie = Katenoide

= Form einer massenbelegten Kette im homogenen Gravitationsfeld



- Minimiere pot. Energie:

$$U = g\int_1^2 dm\, z(x)$$

mit $dm = \rho\, ds \overset{(13.15)}{=} \rho\, dx \sqrt{1+(y')^2}$

$$\longrightarrow \quad U = g\rho \int_{x_1}^{x_2} \underbrace{z(x)\sqrt{1+(z')^2}}_{u}\, dx$$

$$\delta U = 0 \xrightarrow[(13.14)]{} \frac{d}{dx}\frac{\partial u}{\partial z'} - \frac{\partial u}{\partial z} = 0$$

$$\xrightarrow{\text{o.B.}} \quad z z'' - (z')^2 = 1$$

[Hinweis: $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$]

$$\longrightarrow \quad \boxed{z(x) = a \cosh \frac{x-x_0}{a}} \quad (13.6)$$

... Kettenlinie

a?

- Nebenbedingung:

$$L = \int_1^2 ds \quad \text{mit } ds \overset{(13.6)}{=} \dots$$

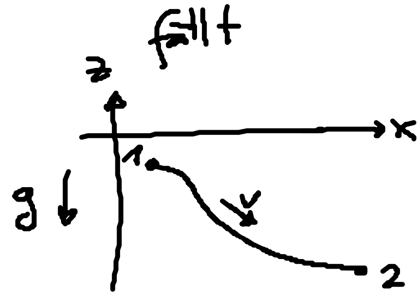
$$= \int_{-\frac{d}{2}}^{\frac{d}{2}} d(x-x_0) \cosh \frac{x-x_0}{a}$$

$$= a \sinh \frac{x-x_0}{a} \Big|_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$\longrightarrow \quad \boxed{\frac{L}{2a} = \sinh \frac{d}{2a}} \quad (13.17)$$

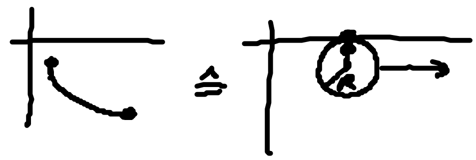
Geg: $L, d \longrightarrow a$!

3. Brachistochronen-Problem: Kurve, auf der ein Teilchen in kürzester Zeit im homog. Grav.feld zwischen 2 Pkten.



Minimiere: $T = \int_1^2 \frac{ds}{v}$ (13.18)

$\longrightarrow$ Teil einer Zykloide



$x = R(\varphi - \sin\varphi)$
$y = R(-1 + \cos\varphi)$

13.3 Euler-Lagrangesche-Gln. 2 Art

• Herleitung aus Hamiltonschem Prinzip

a) kons. Kräften / generalisiertes Potential:

· Lagrangesche Fkt. $L(\{q_j\}, \{\dot{q}_j\}, t) = T - U$

· Wirkung: $S = \int_{t_1}^{t_2} L(\dots)\, dt$

· Variation: $\delta S = \int_{t_1}^{t_2} \delta L(\{q_j\}, \{\dot{q}_j\}, t)\, dt = 0$

wie in Kap. 13.2 für jedes $y(x) \longrightarrow q_j(t)$

$y'(x) \longrightarrow \dot{q}_j(t)$

$x \longrightarrow t$

$\xrightarrow{(13.12)}$ $$\delta S = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right] \delta q_j(t)\, dt = 0$$

δq_j .. unabh. voneinander variierbar $\longrightarrow$ $$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad (13.3)$$

ged.