

English Summary:

O Ornstein-Uhlenbeck process

$$\dot{x} = -kx + \sqrt{2D} \zeta(t) \quad \text{correl. fct. } G(t) = \langle x(t+\tau)x(t) \rangle = \frac{D}{k} e^{-k|\tau|}$$

$$\text{power spectral density } S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(\tau) = \frac{D}{\pi} \frac{1}{\omega^2 + k^2}$$

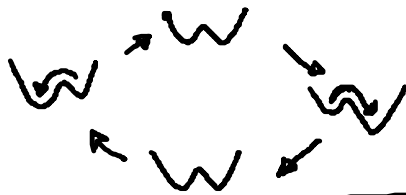
$$\text{direct sol. of Langevin eq.: } x(t) = x(0)e^{-kt} + \sqrt{2D} \int_0^t e^{-k(t-t')} \zeta(t') dt'$$

$$\langle x(t)x(t') \rangle \approx \frac{D}{k} e^{-k|t-t'|} \quad (t, t' \rightarrow \infty)$$

3. Noise-induced oscillations and patterns

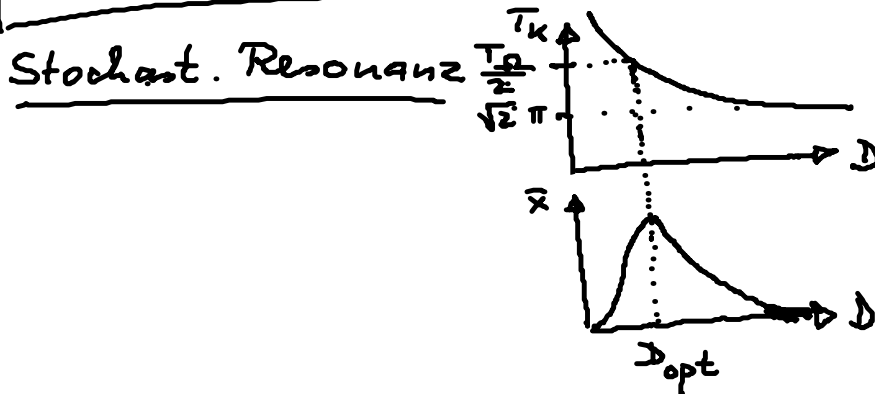
3.1 Stochastic resonance

$$\dot{x} = -V'(x) + A_0 \cos(\Omega t) + \sqrt{2D} \zeta(t) \Rightarrow \langle x(t) \rangle = \bar{x} \cos(\Omega t - \bar{\phi})$$



$$\Omega = \frac{\pi}{\tau_k(D)}$$

Kramers' rate $r_k(D)$



für kleine Amplituden:

$$\bar{x}(D) = \frac{A_0 \langle x^2 \rangle_0}{D} \frac{2r_k(D)}{\sqrt{4r_k^2(D) + \Omega^2}}$$

$$\bar{\phi}(D) = \arctan \frac{\Omega}{2r_k} \quad \text{phase lag}$$

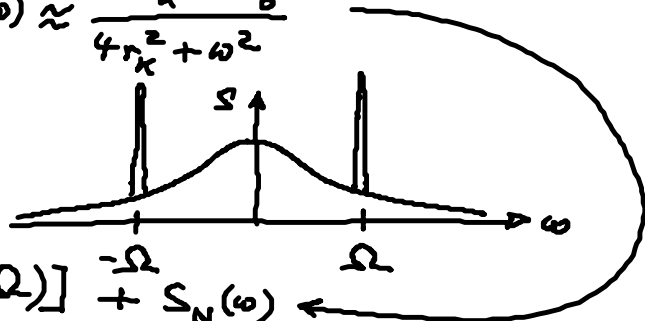
Spektrale Leistungsdichte

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle x(t+\tau)x(t) \rangle \quad \leftarrow \text{zusätzlich über Anfangsphase gemittelt}$$

$$\text{Untergrundrauschen } S_N(\omega) \approx \frac{4r_k \langle x^2 \rangle_0}{4r_k^2 + \omega^2}$$

überlagert durch δ -peaks

bei $\omega = \pm \Omega$

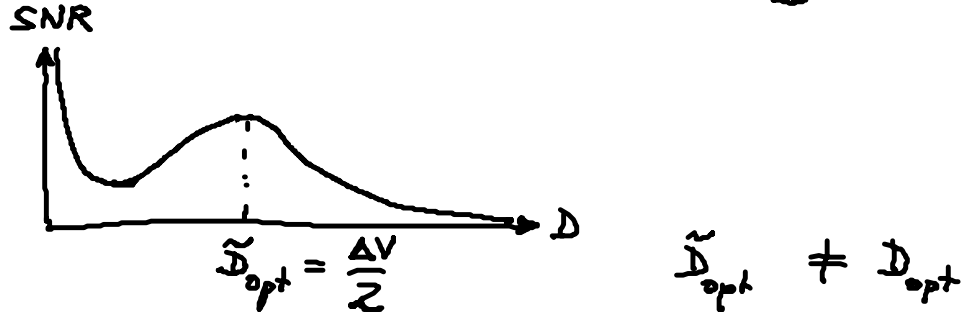


$$S(\omega) = \frac{\pi}{2} \bar{x}(D)^2 [\delta(\omega - \Omega) + \delta(\omega + \Omega)] + S_N(\omega)$$

Signal-to-noise ratio (Maß für Signalverstärkung):

$$\text{SNR} = \frac{2 \lim_{\Delta\omega \rightarrow 0} \int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega) d\omega}{S_N(\Omega)} \sim \frac{e}{D^2}$$

$$\approx \pi \left(\frac{A_0 x_m}{D} \right)^2 r_k(D) \sim \frac{e}{D^2}$$



3.2 Rauschinduzierte Oszillationen

Jetzt: autonome Systeme, ohne externe periodische treibende Kraft

Annahme: determinist. System hat stabilen Fixpt.

⇒ Rauschen kann Oszillationen induzieren

(self-sustained oscillations, stoch. limit cycle)
(LC)

Reviews: Lindner, Garcia-Ojalvo, Neiman, Schimansky-Gier: Effects of noise in excitable systems, Phys. Rep. 392, 321 (2004)

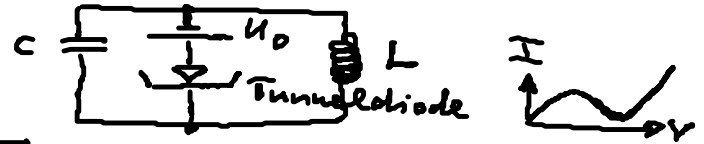
Janson, Balanov, Schöll: Control of noise-induced dynamics. in: Handbook of Chaos Control (Wiley, 2008)

häufig unterhalb einer Bif. eines determin. Grenzzyklus

(Hopf-Bifurk., globale Bifurkation (SNIPER = saddle-node infinite period = SNIC))

1. Beispiel : Van der Pol-Dse. (1920 : nichtlin. el. Stromkreis)

- System knapp unterhalb einer Hopf-Bifurkation



$$\begin{cases} \dot{x} = y \\ \dot{y} = (\varepsilon - x^2)y - \omega_0^2 x + \sqrt{2D} \xi(t) \end{cases}$$

$$\Leftrightarrow \ddot{x} - (\varepsilon - x^2)x + \omega_0^2 x = \tilde{D} \xi(t)$$

nichtlin. Reibung

NB: Rauschintens. $\tilde{D} = \sqrt{2D}$

D=0 (determ.): Fixp. $x^* = y^* = 0$

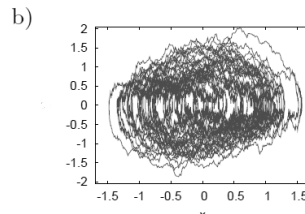
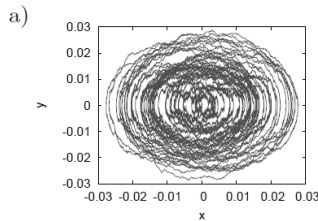
Stab. des Fixp. (linearisiert) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & \varepsilon \end{pmatrix}$

$$\lambda^2 - \lambda \operatorname{tr} A + \det A = 0, \operatorname{tr} A = \varepsilon, \det A = \omega_0^2 > 0$$

- $\Rightarrow \varepsilon = 0$ Hopf-Bif. ($\lambda = \pm i\omega_0$)
- $\varepsilon < 0$ stabiler Fokus
- $\varepsilon > 0$ instab. Fokus + LC

$$\lambda = \frac{\varepsilon}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\varepsilon}{2}\right)^2}$$

Wähle z.B. $\varepsilon = -0.01$, $\omega_0 = 1$
 \Rightarrow rauschinduz. Dse. ($D \neq 0$)



Pomplun et al.
Europhys. Lett.
71, 366 (2005)

Fig. 1 - Simulated phase portraits of noise-induced oscillations of the Van der Pol system for $\varepsilon = -0.01$, $\omega_0 = 1$, $\tau = 0$, $K = 0$: a) $D = 0.003$; b) $D = 0.5$. Integration time: 300 time units.

2. Beispiel : Fitzhugh-Nagumo-Modell
 (Bonhoeffer-van-der-Pol)

- anregbares System (Typ II) : Schwellenverhalten (stabiler Fixp)

Anwendung : Spiking von Neuronen

$$\begin{cases} \varepsilon \dot{x} = x - \frac{x^3}{3} - y \\ \dot{y} = x + a + \tilde{D} \xi(t) \end{cases}$$

Aktivator (schnell)

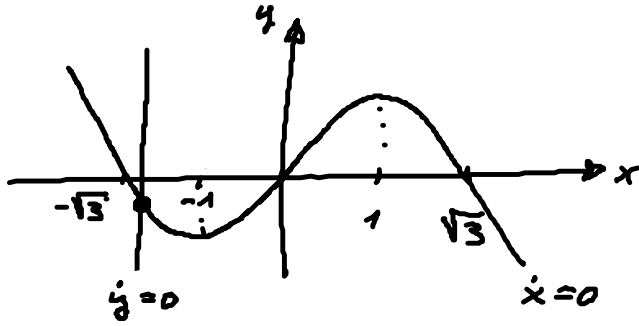
Inhibitor (langsam)

Zeitskalenverh. $\varepsilon \ll 1$

Anreg. Schwelle a

D=0 : Fixp. $x = -a$, $y = -a + \frac{a^3}{3}$

Stab. $\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \frac{1}{\epsilon} \begin{pmatrix} 1-a^2 & -1 \\ \epsilon & 0 \end{pmatrix}$, $t_A = 1-a^2$, $\det A = \epsilon > 0$



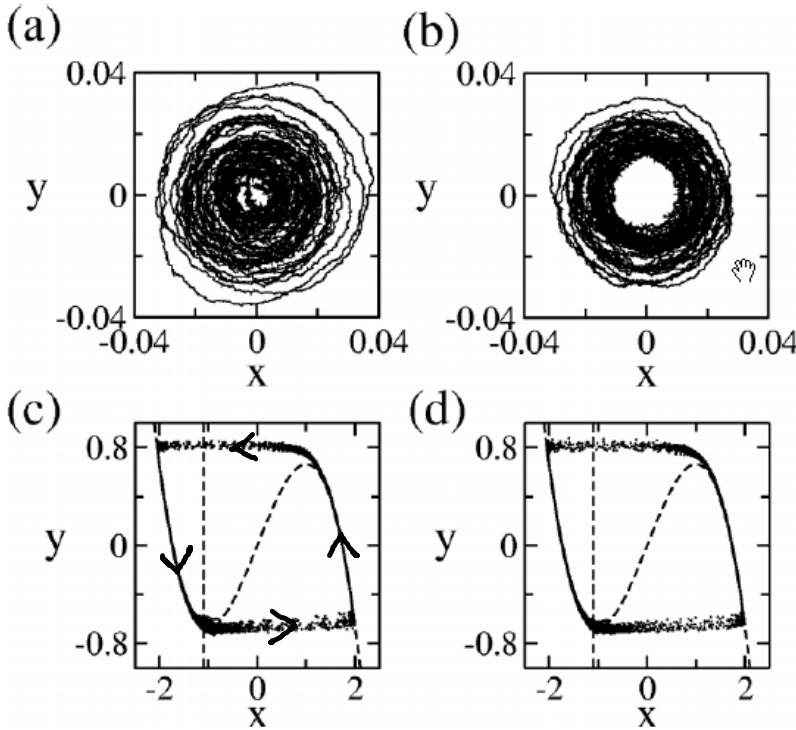
$a = 1$: Hopf-Bif.
 $a < 1$: instab. Fixp. + LC (oszillator. regime)
 $a > 1$: stab. Knoten (anregbares Regime)
 (z.B.: $a = 1.1$, $\epsilon = 0.01$)
 slow-fast-System

$D \neq 0$: rauschinduz. Osz.

Janson, Balanov, Schöll: PRL 93, 010601 (2004)

Balanov, Janson, Schöll: Physics D 199, 1 (2004)

Schöll, Balanov, Janson, Neiman: Stoch. Dyn. 5, 281 (2005)



Janson (2004)

Van der Pol

FitzHugh-Nagumo

FIG. 1. Phase portraits of noise-induced motion: (a),(b) Van der Pol oscillator at $D = 0.003$; (c),(d) FitzHugh-Nagumo system at $D = 0.09$ (the dashed lines denote the null isoclines), (a),(c) $K = 0$; (b),(d) $K = 0.2$, $\tau = T_0$.

3.3 Kohärenzresonanz

Gang, Ditinger, Ning, Haken: Stoch. Reson. without external periodic forcing, PRL 71, 807 (1993)

Pikovsky, Kurths: Coherence resonance in a noise-driven excitable system, PRL 78, 775 (1997)

Neiman, Sapsis, Stone: PRE 56, 270 (1997)

- konstruktiver Einfluss von Rauschen

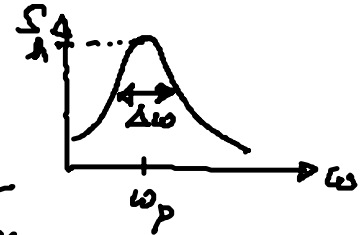
• Regularität („Kohärenz“) der rauschinduz. Osz.
am größten für endl. Rauschintensität D_{opt}

• Maß für Regularität:

- Signal-Rausch-Verhältnis (SNR)
(Haken)

$$\beta = \frac{k}{\Delta\omega/\omega_p}$$

quality factor



- Standardabweich. der ISI

(interspike interval) 

$$R_T = \frac{\sqrt{\langle T_{ISI}^2 \rangle - \langle T_{ISI} \rangle^2}}{\langle T_{ISI} \rangle}$$

(normierte ISI-Flukt.)

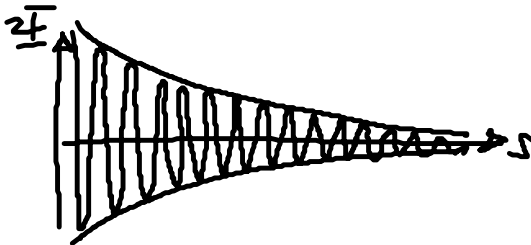
- Korrelationszeit $t_{cor} := \frac{1}{\underline{\mathbb{F}}(0)} \int_0^\infty |\underline{\mathbb{F}}(s)| ds$

(Autokorrel. fkt. $\underline{\mathbb{F}}(s) := \langle [x(t+s) - \langle x \rangle][x(t) - \langle x \rangle] \rangle$)
Varianz $\underline{\mathbb{F}}(0) = \sigma^2$

Motivation der Def.:

für lin. stoch. Prozesse $\dot{x} = -(k+i\omega_0)x + \xi(t)$:

$$\underline{\mathbb{F}}(s) = \underline{\mathbb{F}}(0) e^{-ks} \cos(\omega_0 s) \quad \left(\begin{array}{l} \text{\S 2.3. stat. Ornstein-Ull.} \\ \text{Re } e^{-(k+i\omega_0)s} \end{array} \right)$$



exp. abklingende Enveloppe

Trägerfrequ. ω_0

$k > 0$ (stabiler Fixp.)

Zus.hang zwischen t_{cor} und k :

$$t_{cor} = \int_0^{\infty} e^{-ks} |\cos \omega_0 s| ds$$

Approx. $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos \phi d\phi = \frac{2}{\pi}$ für $k \ll \omega_0$ (Füllfaktor)

$$\Rightarrow t_{cor} \approx \frac{2}{\pi} \int_0^{\infty} e^{-ks} ds = \frac{2}{\pi k}$$

also $\underline{z}(s) = \underline{z}(0) e^{-\frac{2}{\pi} \frac{s}{t_{cor}}} \cos(\omega_0 s)$

$k = |\operatorname{Re}(\text{Eigenwert des Fixp.})| = \text{Bifurk.par.}$
 $\operatorname{Re} \lambda < 0$ (Abstand von Hopf-Bif.)

\Rightarrow je stabiler der Fixp., umso kürzer die Korrel.zeit
(je weiter weg von der Hopf-Bif.)