English Summary:

Ornstein-Uhlenbeck process

$\boxed{\dot{x} = -kx + \sqrt{2D}\,\xi(t)}$ correl. fct. $G(\tau) = \langle x(t+\tau)\,x(t)\rangle = \frac{D}{k}\,e^{-k|\tau|}$

power spectral density $S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, G(\tau) = \frac{D}{\pi}\,\frac{1}{\omega^2+k^2}$

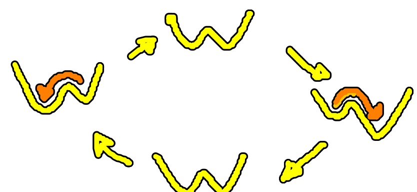
direct sol. of Langevin eq.: $x(t) = x(0)\,e^{-kt} + \sqrt{2D}\int_0^t e^{-k(t-t')}\,\xi(t')\,dt'$

$\langle x(t)\,x(t')\rangle \approx \frac{D}{k}\,e^{-k|t-t'|} \quad (t, t' \to \infty)$

3. Noise-induced oscillations and patterns

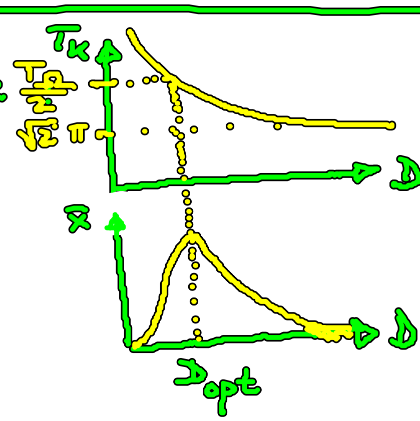
3.1 Stochastic resonance

$\dot{x} = -V'(x) + A_0\cos(\Omega t) + \sqrt{2D}\,\xi(t) \Rightarrow \langle x(t)\rangle = \bar{x}\cos(\Omega t - \bar{\phi})$



$\boxed{\Omega = \frac{\pi}{r_K(D)}}$ Kramers' rate $r_K(D)$

Stochast. Resonanz



für kleine Amplituden:

$$\bar{x}(D) = \frac{A_0\langle x^2\rangle_0}{D}\,\frac{2r_K(D)}{\sqrt{4r_K^2(D)+\Omega^2}}$$

$$\bar{\phi}(D) = \arctan\frac{\Omega}{2r_K} \quad \text{phase lag}$$

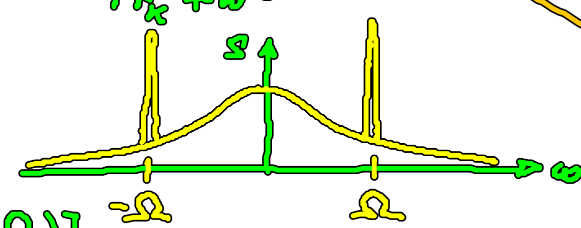
Spektrale Leistungsdichte

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\,\langle x(t+\tau)\,x(t)\rangle$$

zusätzlich über Anfangsphase gemittelt

Untergrundrauschen $S_N(\omega) \approx \frac{4r_K\langle x^2\rangle_0}{4r_K^2+\omega^2}$

überlagert durch δ-Peaks bei $\omega = \pm\Omega$

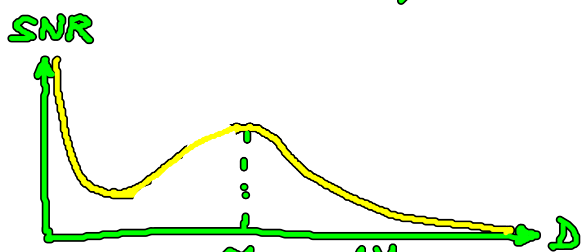


$$S(\omega) = \frac{\pi}{2}\,\bar{x}(D)^2\,[\delta(\omega-\Omega)+\delta(\omega+\Omega)] + S_N(\omega)$$

Signal-to-noise ratio (Maß für Signalverstärkung):

$$\mathrm{SNR} = \frac{2 \lim\limits_{\Delta\omega \to 0} \int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega)\, d\omega}{S_N(\Omega)}$$

$$\approx \pi \left(\frac{A_0 x_m}{D} \right)^2 r_K(D) \sim \frac{e^{-\frac{\Delta V}{D}}}{D^2}$$



$$\tilde{D}_{opt} = \frac{\Delta V}{2} \qquad \tilde{D}_{opt} \neq D_{opt}$$

3.2 Rauschinduzierte Oszillationen

Jetzt: autonome Systeme, ohne externe periodische treibende Kraft

Annahme: determinist. System hat stabilen Fixpkt.

$\Rightarrow$ Rauschen kann Oszillationen induzieren

(self-sustained oscillations, stoch. limit cycle (LC))

Reviews: Lindner, Garcia-Ojalvo, Neiman, Schimansky-Geier: Effects of noise in excitable systems, Phys. Rep. 392, 321 (2004)

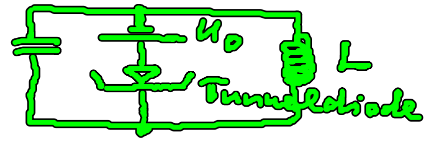
Janson, Balanov, Schöll: Control of noise-induced dynamics. in: Handbook of Chaos Control (Wiley, 2008)

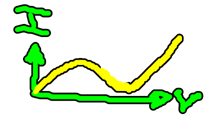
häufig unterhalb einer Bif. eines determ. Grenzzyklus

(Hopf-Bifurk., globale Bifurkation (SNIPER = saddle-node infinite period = SNIC))

1. Beispiel : Van der Pol-Osz. (1920 : nichtlin. el. Stromkreis)

- System knapp unterhalb einer Hopf-Bifurkation





$$\boxed{\begin{aligned} \dot x &= y \\ \dot y &= (\varepsilon - x^2)y - \omega_0^2 x + \sqrt{2D}\,\xi(t) \end{aligned}}$$

$$\Leftrightarrow \ddot x \underbrace{- (\varepsilon - x^2)\dot x}_{\text{nichtlin. Reibung}} + \omega_0^2 x = \tilde D\, \xi(t)$$

NB: Rauschintens. $\tilde D = \sqrt{2D}$

D = 0 (determ.) : Fixp. $x^* = y^* = 0$

Stab. des Fixp. (linearisiert) $\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & \varepsilon \end{pmatrix}$

$\lambda^2 - \lambda\, \mathrm{tr} A + \det A = 0$, $\mathrm{tr} A = \varepsilon$, $\det A = \omega_0^2 > 0$

$$\left.\begin{aligned} \Rightarrow\ &\varepsilon = 0 \quad \text{Hopf-Bif. } (\lambda = \pm i\omega_0) \\ &\varepsilon < 0 \quad \text{stabiler Fokus} \\ &\varepsilon > 0 \quad \text{instab. Fokus + LC} \end{aligned}\right\} \quad \lambda = \frac{\varepsilon}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\varepsilon}{2}\right)^2}$$

Wähle z.B. $\varepsilon = -0.01$, $\omega_0 = 1$

$\Rightarrow$ rauschinduz. Osz. ($D \neq 0$)

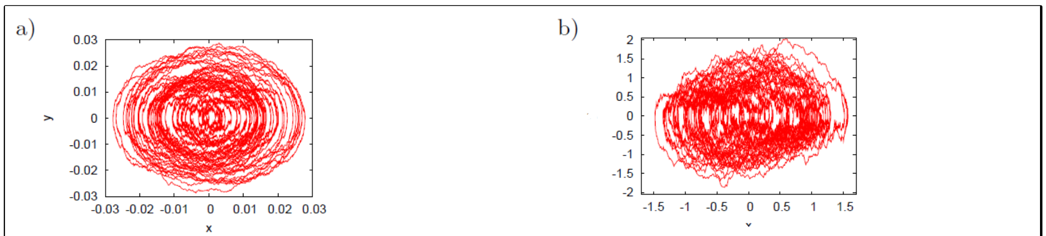


Fig. 1 – Simulated phase portraits of noise-induced oscillations of the Van der Pol system for $\varepsilon = -0.01$, $\omega_0 = 1$, $\tau = 0$, $K = 0$: a) $D = 0.003$; b) $D = 0.5$. Integration time: 300 time units.

Pomplun et al. Europhys. Lett. 71, 366 (2005)

2. Beispiel : FitzHugh-Nagumo-Modell (Bonhoeffer-van-der Pol)

- anregbares System (Typ II) : Schwellenverhalten (stabiler Fixp)

Anwendung : Spiking von Neuronen

$$\boxed{\begin{aligned} \varepsilon \dot x &= x - \frac{x^3}{3} - y \\ \dot y &= x + a \qquad + \tilde D \xi(t) \end{aligned}}$$

Aktivator (schnell)

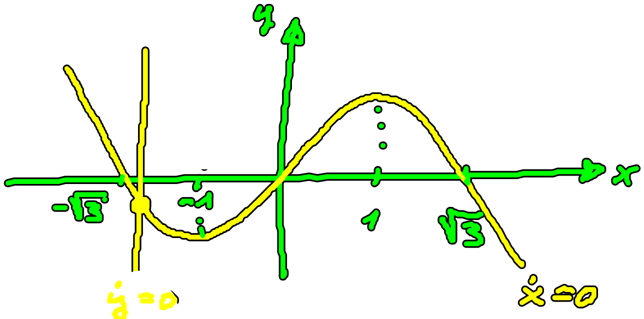
Inhibitor (langsam)

Zeitskalen sep. $\varepsilon \ll 1$

Anreg. schwelle a

D = 0 : Fixp. $x = -a$, $y = -a + \frac{a^3}{3}$

Stab. $\begin{pmatrix}\delta\dot x\\ \delta\dot y\end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix}1-a^2 & -1\\ \varepsilon & 0\end{pmatrix}$, $\mathrm{tr}\,A = 1-a^2$, $\det A = \varepsilon > 0$



$a = 1$: Hopf-Bif.

$a < 1$: instab. Fixp. + LC (oszillator. Regime)

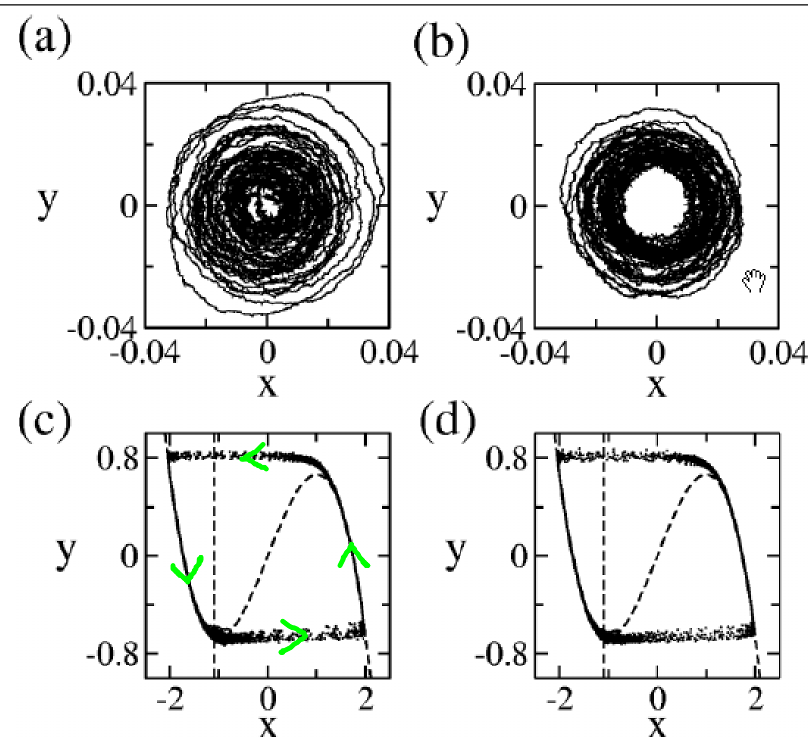
$a > 1$: stab. Knoten (anregbares Regime)

(z.B.: $a = 1.1$, $\varepsilon = 0.01$) slow-fast-System

$D \neq 0$: rauschinduz. Osz.

Janson, Balanov, Schöll: PRL 93, 010601 (2004)
Balanov, Janson, Schöll: Physica D 199, 1 (2004)
Schöll, Balanov, Janson, Neiman: Stoch. Dyn. 5, 281 (2005)

Janson (2004)



Van der Pol

FitzHugh-Nagumo

FIG. 1. Phase portraits of noise-induced motion: (a),(b) Van der Pol oscillator at $D = 0.003$; (c),(d) FitzHugh-Nagumo system at $D = 0.09$ (the dashed lines denote the null isoclines), (a),(c) $K = 0$; (b),(d) $K = 0.2$, $\tau = T_0$.

3.3 Kohärenzresonanz

Gang, Ditzinger, Ning, Haken: Stoch. reson. without external periodic forcing, PRL 71, 807 (1993)

Pikovsky, Kurths: Coherence resonance in a noise-driven excitable system, PRL <u>78</u>, 775 (1997)

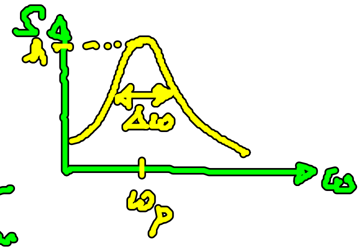
Neiman, Saparin, Stone: PRE <u>56</u>, 270 (1997)

— konstruktiver Einfluss von Rauschen

- Regularität („Kohärenz") der rauschinduz. Osz. am größten für <u>endl.</u> Rauschintensität D_{opt}

- Maß für Regularität:

  – <u>Signal-Rausch-Verhältnis</u> (SNR) (Haken) $\beta = \underbrace{\frac{h}{\Delta\omega/\omega_p}}_{\text{quality factor}}$



  – <u>Standardabweich. der ISI (interspike interval)</u>

$$R_T = \frac{\sqrt{\langle T_{ISI}^2\rangle - \langle T_{ISI}\rangle^2}}{\langle T_{ISI}\rangle} \qquad \text{(normierte ISI-Flukt.)}$$

  – Korrelationszeit $t_{cor} := \frac{1}{\Psi(0)}\int_0^\infty |\Psi(s)|\, ds$

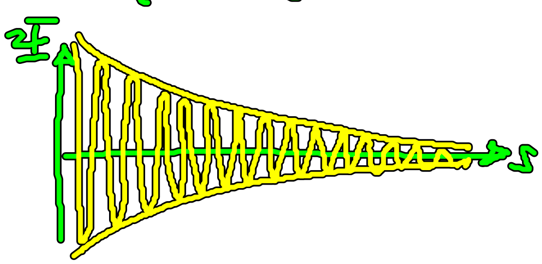
(Autokorrel. fkt. $\Psi(s) := \langle [x(t+s)-\langle x\rangle][x(t)-\langle x\rangle]\rangle$)
Varianz $\Psi(0) = \sigma^2$

<u>Motivation der Def.</u>:

für <u>lin.</u> stoch. Prozesse $\dot{x} = -(k+i\omega_0)x + \xi(t)$:

$$\Psi(s) = \Psi(0)\, e^{-ks}\cos(\omega_0 s)$$

(§2.3.: stat. Ornstein-Uhl.
Re $e^{-(k+i\omega_0)s}$)



exp. abklingende Enveloppe
Trägerfrequ. ω_0
$k > 0$ (<u>stabiler</u> Fixp.)

Zus.hang zwischen t_{cor} und k:

$$t_{cor} = \int_0^\infty e^{-ks} |\cos \omega_0 s| \, ds$$

Approx. $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos\phi \, d\phi = \frac{2}{\pi}$ für $k \ll \omega_0$ (Füllfaktor)

$$\Rightarrow \boxed{t_{cor} \approx \frac{2}{\pi} \int_0^\infty e^{-ks} ds = \frac{2}{\pi k}}$$

also $\Phi(s) = \Phi(0)\, e^{-\frac{2}{\pi}\frac{s}{t_{cor}}} \cos(\omega_0 s)$

$k = |\mathrm{Re}(\text{Eigenwert des Fixp.})| = $ Bifurk.par.
$\mathrm{Re}\,\lambda < 0$ (Abstand von Hopf-Bif.)

$\rightarrow$ je stabiler der Fixp., umso kürzer die Korrel.zeit
(je weiter weg von der Hopf-Bif.)