English Summary:

classical Van der Pol osc.: $\ddot{Q} + \omega_0^2 Q - \varepsilon(1 - 2Q^2)\dot{Q} = 0 \qquad \varepsilon > 0$

neg. lin. damping $\quad$ pos. nonlin. damping

network of VdP osc. (nonlocal coupling of rang R):

$\dot{x}_k = F(x_k) + \frac{\sigma}{2R} \sum_{j=k-R}^{k+R} H(x_j - x_k)$, $k = 1, \dots, N$, $x_k = \binom{Q}{P} \in \mathbb{R}^2$, $P = \dot{Q}$

H 2x2 coupling matrix

$\Rightarrow$ chimera states

$\varepsilon \ll 1$: $\dot{\alpha} = \frac{\varepsilon}{2}(1 - |\alpha|^2)\alpha$ $\quad$ Stuart-Landau osc. $\alpha = \frac{1}{\sqrt{2}}(Q + iP)$

Quantum Van der Pol osc.:

$\dot{\rho}_S = 2\kappa_1 \left(a^+ \rho_S a - \frac{1}{2}\{\rho_S, a a^+\}\right) + 2\kappa_2 \left(a^2 \rho_S (a^+)^2 - \frac{1}{2}\{\rho_S, (a^+)^2 a^2\}\right)$

1-photon $\qquad$ 2-photon

4.7.2 Quantum Van der Pol oscillator

Bastidas, Omelchenko, Zakharova, Schöll, Brandes, PRE 92, 062924 (28.12.2015)

—"— , in "Control of Self-Organizing Nonlinear Systems" (eds. Schöll, Klapp, Hövel), Springer 2016

Aim: Lindblad Master eq. with dissipation rates κ_1 (linear neg. damping) and κ_2 (nonlinear loss)

Decomposition of bosonic operators

$$\boxed{a(t) = \underbrace{\tilde{a}}_{\text{quantum fluctuations}} + \underbrace{\alpha(t)}_{\text{mean field}}} \qquad a^+(t) = \tilde{a}^+ + \alpha^*(t)$$

quantum fluctuations: annih. op. $\tilde{a}$

mean field: $\alpha = \langle\alpha| a |\alpha\rangle \in \mathbb{C}$, semiclass. trajectory

coherent state α: $a|\alpha\rangle = \alpha|\alpha\rangle$

Quantum fluctuations about a semiclassical trajectory $\alpha(t)$:

Density matrix in the co-moving frame (co-moving with $\alpha(t)$):

$$\rho_\alpha(t) = D^+(\alpha)\,\rho(t)\,D(\alpha) \qquad \rho \equiv \rho_s$$

with displacement op. (unitary) $D(\alpha) = e^{\alpha a^+ - \alpha^* a} = e^{\alpha\tilde{a}^+ - \alpha^*\tilde{a}}$

$$= e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha\tilde{a}^+}\, e^{-\alpha^*\tilde{a}}$$

$$\dot{\rho}_\alpha(t) = -\frac{i}{\hbar}\left[H^{(\alpha)}(t), \rho_\alpha(t)\right] + 2\kappa_1 D^+(\alpha)\left(\tilde{a}^+\rho\,\tilde{a} - \frac{1}{2}\{\rho, \tilde{a}\tilde{a}^+\}\right)D(\alpha)$$

$$+2\kappa_2 D^+(\alpha)\left(\tilde{a}^2\rho(\tilde{a}^+)^2 - \frac{1}{2}\{\rho, (\tilde{a}^+)^2\tilde{a}^2\}\right)D(\alpha)$$

$$= -\frac{i}{\hbar}\left[H^{(\alpha)}(t), \rho_\alpha(t)\right] + \hat{\mathcal{L}}_1\rho_\alpha + \hat{\mathcal{L}}_2\rho_\alpha$$

Liouville operators of dissipation

with $H^{(\alpha)}(t) := -i\hbar D^+(\alpha)\,\partial_t D(\alpha)$

$$= -\frac{i\hbar}{2}\left[\dot{\alpha}(t)\alpha^*(t) - \alpha(t)\dot{\alpha}^*(t)\right]$$

$$-\, i\hbar\left[\dot{\alpha}(t)\,\tilde{a}^+ - \dot{\alpha}^*(t)\,\tilde{a}\right]$$

gaussian quantum fluctuations for $|\alpha(t)| \gg 1$: (semiclassical high-density limit)

$\Rightarrow$ neglect all non-gaussian (non-quadratic) fluctuations

$$\hat{\mathcal{L}}_1\rho_\alpha = \underbrace{2\kappa_1\left(\tilde{a}^+\rho_\alpha\tilde{a} - \frac{1}{2}\{\rho_\alpha, \tilde{a}\tilde{a}^+\}\right)}_{\text{incoherent (dissip.) part}} \underbrace{\cancel{-\frac{i}{\hbar}\left[i\hbar\kappa_1\alpha\tilde{a}^+, \rho_\alpha\right]} \cancel{-\frac{i}{\hbar}\left[-i\hbar\kappa_1\alpha^*\tilde{a}, \rho_\alpha\right]}}_{\text{coherent part}}$$

lin. in $\tilde{a}^+$ or $\tilde{a}$

$\hat{\mathcal{L}}_2\rho_\alpha$ analogously

$$\Rightarrow \quad (1) \quad \boxed{\begin{aligned}\dot{\rho}_\alpha(t) = &-i\kappa_2\left[i(\alpha^*)^2\tilde{a}^2 - i\alpha^2(\tilde{a}^+)^2, \rho_\alpha\right] \\ &+ 2\kappa_1\left(\tilde{a}^+\rho_\alpha\tilde{a} - \frac{1}{2}\{\rho_\alpha, \tilde{a}\tilde{a}^+\}\right) + 8\kappa_2|\alpha|^2\left(\tilde{a}\rho_\alpha\tilde{a}^+ - \frac{1}{2}\{\rho_\alpha, \tilde{a}^+\tilde{a}\}\right)\end{aligned}}$$

position of co-moving frame:

$$(2) \quad \boxed{\dot{\alpha}(t) = \kappa_1\alpha(t) - 2\kappa_2\alpha(t)|\alpha(t)|^2}$$

semiclassical trajectory

Interpretation:

initially coherent state $\varrho(t=0) = |\alpha(0)\rangle\langle\alpha(0)|$

$$\Leftrightarrow \quad \varrho_\alpha(t=0) = |0\rangle\langle 0|$$ vacuum state in co-moving frame centered at $\alpha(0)$

$\hat{=}$ initial cond. for (2)

sol. of semiclassical Eq. (2) $\triangleq$ classical Stuart-Landau eq.

$$\dot{\alpha} = \frac{\varepsilon}{2}(1 - |\alpha|^2)\alpha$$

with $\kappa_1 = \frac{\varepsilon}{2}$, $\kappa_2 = \frac{\varepsilon}{4}$

$\Rightarrow$ (i) limit cycle: $\boxed{|\alpha|^2 = \frac{\kappa_1}{2\kappa_2}}$

(ii) time-dependent squeezing described by master eq. (1)

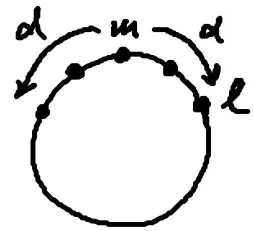
4.7.3 Network of quantum Van der Pol osc.

Ring network of N Van der Pol osc. a_ℓ, a_ℓ^+, $\ell = 1, \dots, N$
indices mod N

$$(3) \quad \boxed{\dot{\varrho} = -\frac{i}{\hbar}[H, \varrho] + 2\sum_{\ell=1}^{N}\left[\kappa_1 \mathcal{D}(a_\ell^+) + \kappa_2 \mathcal{D}(a_\ell^2)\right]}$$

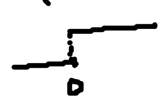
with dissipative processes $\mathcal{D}(\hat{O}) := \hat{O}\varrho\hat{O}^+ - \frac{1}{2}\{\varrho, \hat{O}^+\hat{O}\}$

(new!)
nonlocal coupling of range d:



$$H = \hbar \sum_{\substack{m=1 \\ m \neq \ell}}^{N} K_{\ell m}(a_\ell^+ a_m + a_\ell a_m^+)$$

coupling matrix $K_{\ell m} = \frac{V}{2d}\,\Theta(d - |\ell - m|)$



coupling strength V

general form of Lindblad master eq.

$$\dot{\varrho} = -\frac{i}{\hbar}[H, \varrho] + \sum_\mu \gamma_\mu \left(L_\mu \varrho L_\mu^+ - \frac{1}{2}\{\varrho, L_\mu^+ L_\mu\}\right)$$

Lindblad op.

eff. Hamiltonian

$$H_{eff} := H - \frac{i\hbar}{2} \sum_\mu \gamma_\mu L_\mu^+ L_\mu$$

$$(3) \Rightarrow H_{eff} = \underbrace{i\hbar\kappa_1 \sum_{\ell=1}^{N} (a_\ell^+ a_\ell + 1)}_{\text{diss.}} - \underbrace{i\hbar\kappa_2 \sum_{\ell=1}^{N} \hat{n}_\ell (\hat{n}_\ell - 1)}_{\text{diss.}} \qquad \hat{n}_\ell := a_\ell^+ a_\ell$$

$$+ \hbar \sum_{\substack{m=1 \\ m \neq \ell}}^{N} K_{\ell m} (a_\ell^+ a_m + a_\ell a_m^+)$$

(nonlocal coupling)

Bose-Hubbard model with long-range interaction and complex on-site self-energies and chem. pot.

($\Rightarrow$ application to driven dissipation Bose-Einstein condensation)

<u>gaussian quantum fluctuations</u>

expansion $b_\ell(t) := D^+(\alpha(t)) a_\ell D(\alpha(t)) = \underbrace{\tilde{a}_\ell}_{\text{quantum fluct.}} + \underbrace{\alpha_\ell(t)}_{\text{mean-field}}$

with displacement op $D(\alpha) = e^{\alpha \tilde{a}^+ - \alpha^* \tilde{a}}$

$$\alpha(t) \equiv (\alpha_1, \alpha_2, \dots, \alpha_N)$$
$$\tilde{a} \equiv (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N)$$

<u>Semiclassical regime</u> $|\alpha_\ell(t)| \gg \underbrace{\|\tilde{a}_\ell\|}_{\substack{\text{quantum fluct.} \\ \text{gaussian}}}$

expansion as Eq.(1) $\Longrightarrow$ Master eq. in co-moving frame, describes <u>quantum fluctuations</u>

$$\Rightarrow (4) \quad \boxed{\dot{\varrho}_\alpha \simeq \underbrace{-\frac{i}{\hbar} [H_Q^{(\alpha)}, \varrho_\alpha]}_{\text{coherent dyn.}} + 2 \sum_{\ell=1}^{N} [\kappa_1 \mathcal{D}(\tilde{a}_\ell^+) + 4\kappa_2 |\alpha_\ell|^2 \mathcal{D}(\tilde{a}_\ell)]}$$

$$\Downarrow$$

$$H_Q^{(\alpha)}(t) = \hbar \sum_{\ell=1}^{N} \Big(i\kappa_2 (\alpha_\ell^*)^2 \tilde{a}_\ell^2 - i\kappa_2 \alpha_\ell^2 (\tilde{a}_\ell^+)^2 + \sum_{n=1}^{N} K_{\ell\, \ell+n} (\tilde{a}_\ell^+ \tilde{a}_{\ell+n} + \tilde{a}_\ell \tilde{a}_{\ell+n}^+) \Big)$$

$$+ \hbar \sum_{\ell=1}^{N} \left(-i [\dot{\alpha}_\ell \tilde{a}_\ell^+ - \dot{\alpha}_\ell^* a_\ell] + i\kappa_1 \alpha_\ell \tilde{a}_\ell^+ - i\kappa_1 \alpha_\ell^* \tilde{a}_\ell \right)$$

$$+ \hbar \sum_{\ell=1}^{N} \left(2i\kappa_2 \alpha_\ell (\alpha_\ell^*)^2 \tilde{a}_\ell - 2i\kappa_2 \alpha_\ell^* \alpha_\ell^2 \tilde{a}_\ell^+ \right)$$

$$+\hbar \sum_{r=1}^{N} K_{l,l+r} \left(\alpha_{l+r} \tilde{a}_l^+ + \alpha_{l+r}^* \tilde{a}_l + \alpha_l^* \tilde{a}_{l+r} + \alpha_l \tilde{a}_{l+r}^+ \right)$$

semiclass. trajectory (⇒ lin. terms in Master eq. vanish):

$$(5) \quad \boxed{\dot{\alpha}_l(t) = \alpha_l(t)\left(\kappa_1 - 2\kappa_2 |\alpha_l(t)|^2\right) - i \sum_{\substack{s=1 \\ s\neq l}}^{N} K_{ls}\alpha_s(t) \;.}$$

$$(4) \Rightarrow \frac{d}{dt}\langle \tilde{a}_i \rangle = \mathrm{tr}\left[\tilde{a}_i \dot{\varrho}_\alpha(t)\right] = \kappa_1 \langle \tilde{a}_i \rangle - 4\kappa_2 |\alpha_i|^2 \langle \tilde{a}_i \rangle$$

$$- 2\kappa_2 \alpha_i^2 \langle \tilde{a}_i^+ \rangle$$

$$- i \sum_{\substack{s=1 \\ s\neq i}}^{N} K_{is} \langle \tilde{a}_s \rangle$$