English Summary:

<u>classical Van der Pol osc.</u>: $\ddot{Q} + \omega_0^2 Q - \varepsilon(1 - 2Q^2)\dot{Q} = 0$ $\quad \varepsilon > 0$

neg. lin. damping $\quad$ pos. nonlin. damping

network of VdP osc. (nonlocal coupling of rang R):

$$\dot{x}_k = F(x_k) + \frac{\sigma}{2R}\sum_{j=k-R}^{k+R} H(x_j - x_k), \quad k = 1, \dots, N, \quad x_k = \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^2, \quad P = \dot{Q}$$

H 2x2 coupling matrix

$\rightarrow$ chimera states

$\varepsilon \ll 1$: $\dot{\alpha} = \frac{\varepsilon}{2}(1 - |\alpha|^2)\alpha$ $\quad$ Stuart-Landau osc. $\alpha = \frac{1}{\sqrt{2}}(Q + iP)$

<u>Quantum Van der Pol osc.</u>:

$$\dot{\rho}_s = \underbrace{2\kappa_1\left(a^\dagger \rho_s a - \frac{1}{2}\{\rho_s, a a^\dagger\}\right)}_{\text{1-photon}} + \underbrace{2\kappa_2\left(a^2 \rho_s (a^\dagger)^2 - \frac{1}{2}\{\rho_s, (a^\dagger)^2 a^2\}\right)}_{\text{2-photon}}$$

4.7.2 Quantum Van der Pol oscillator

Bastidas, Omelchenko, Zakharova, Schöll, Brandes, PRE 92, 062924 (28.12.2015)

—"— , in "Control of Self-Organizing Nonlinear Systems" (eds. Schöll, Klapp, Hövel), Springer 2016

<u>Aim</u>: Lindblad Master eq. with dissipation rates κ_1 (linear neg. damping) and κ_2 (nonlinear loss)

<u>Decomposition of bosonic operators</u>

$$\boxed{a(t) = \underbrace{\tilde{a}}_{\substack{\text{quantum fluctuations} \\ \text{annih. op. } \tilde{a}}} + \underbrace{\alpha(t)}_{\text{mean field}}} \qquad a^\dagger(t) = \tilde{a}^\dagger + \alpha^*(t)$$

$\alpha = \langle\alpha| a |\alpha\rangle \in \mathbb{C}$ $\quad$ coherent state α

semiclass. trajectory $\quad$ $a|\alpha\rangle = \alpha|\alpha\rangle$

Quantum fluctuations about a semiclassical trajectory $\alpha(t)$:

Density matrix in the co-moving frame (co-moving with $\alpha(t)$):

$$\rho_\alpha(t) = D^+(\alpha)\,\rho(t)\,D(\alpha) \qquad \rho = \rho_S$$

with displacement op. $D(\alpha) = e^{\alpha a^+ - \alpha^* a} = e^{\alpha \tilde{a}^+ - \alpha^* \tilde{a}}$
(unitary)

$$= e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha \tilde{a}^+}\, e^{-\alpha^* \tilde{a}}$$

$$\dot{\rho}_\alpha(t) = -\frac{i}{\hbar}\left[H^{(\alpha)}(t), \rho_\alpha(t)\right] + 2\kappa_1 D^+(\alpha)\left(\tilde{a}^+ \rho\, \tilde{a} - \frac{1}{2}\{\rho, \tilde{a}\tilde{a}^+\}\right) D(\alpha)$$

$$+ 2\kappa_2 D^+(\alpha)\left(\tilde{a}^2 \rho (\tilde{a}^+)^2 - \frac{1}{2}\left\{\rho, (\tilde{a}^+)^2 \tilde{a}^2\right\}\right) D(\alpha)$$

$$= -\frac{i}{\hbar}\left[H^{(\alpha)}(t), \rho_\alpha(t)\right] + \hat{\mathcal{L}}_1 \rho_\alpha + \hat{\mathcal{L}}_2 \rho_\alpha$$

Liouville operators of dissipation

with $H^{(\alpha)}(t) := -i\hbar D^+(\alpha)\, \partial_t D(\alpha)$

$$= -\frac{i\hbar}{2}\left[\dot{\alpha}(t)\alpha^*(t) - \alpha(t)\dot{\alpha}^*(t)\right]$$

$$- i\hbar\left[\dot{\alpha}(t)\,\tilde{a}^+ - \dot{\alpha}^*(t)\,\tilde{a}\right]$$

<u>gaussian</u> quantum fluctuations for $|\alpha(t)| \gg 1$:
(semiclassical high-density limit)

$\Rightarrow$ neglect all non-gaussian (non-quadratic) fluctuations

$$\hat{\mathcal{L}}_1 \rho_\alpha = 2\kappa_1\left(\tilde{a}^+ \rho_\alpha \tilde{a} - \frac{1}{2}\{\rho_\alpha, \tilde{a}\tilde{a}^+\}\right) \cancel{- \frac{i}{\hbar}\left[i\hbar\kappa_1\alpha\tilde{a}^+, \rho_\alpha\right]}$$

incoherent (dissip.) part

$$\cancel{- \frac{i}{\hbar}\left[-i\hbar\kappa_1\alpha^*\tilde{a}, \rho_\alpha\right]}$$

coherent part

lin. in $\tilde{a}^+$ or $\tilde{a}$

$\hat{\mathcal{L}}_2 \rho_\alpha$ analogously

$$\Rightarrow \quad (1) \quad \boxed{\begin{aligned} \dot{\rho}_\alpha(t) &= -i\kappa_2\left[i(\alpha^*)^2 \tilde{a}^2 - i\alpha^2 (\tilde{a}^+)^2, \rho_\alpha\right] \\ &\quad + 2\kappa_1\left(\tilde{a}^+ \rho_\alpha \tilde{a} - \frac{1}{2}\{\rho_\alpha, \tilde{a}\tilde{a}^+\}\right) + 8\kappa_2|\alpha|^2\left(\tilde{a}\rho_\alpha \tilde{a}^+ - \frac{1}{2}\{\rho_\alpha, \tilde{a}^+\tilde{a}\}\right) \end{aligned}}$$

position of co-moving frame:

$$(2) \quad \boxed{\dot{\alpha}(t) = \kappa_1 \alpha(t) - 2\kappa_2 \alpha(t)|\alpha(t)|^2}$$

semiclassical trajectory

Interpretation:

initially coherent state $\rho(t=0) = |\alpha(0)\rangle\langle\alpha(0)|$

$\Leftrightarrow \quad \rho_\alpha(t=0) = |0\rangle\langle 0|$ vacuum state in co-moving frame centered at $\alpha(0)$

$\hat{=}$ initial cond. for (2)

Sol. of semiclassical Eq.(2) $\hat{=}$ classical Stuart-Landau eq.

$$\dot{\alpha} = \frac{\varepsilon}{2}\left(1 - |\alpha|^2\right)\alpha$$

with $\kappa_1 = \frac{\varepsilon}{2}$, $\kappa_2 = \frac{\varepsilon}{4}$

$\Rightarrow$ (i) limit cycle: $\boxed{|\alpha|^2 = \frac{\kappa_1}{2\kappa_2}}$

(ii) time-dependent squeezing described by master eq. (1)

4.7.3 Network of quantum Van der Pol osc.

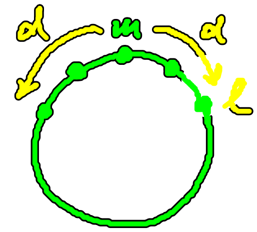
Ring network of N Van der Pol osc. a_ℓ, a_ℓ^+, $\ell = 1, \dots, N$ indices mod N

$$(3) \quad \boxed{\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + 2\sum_{\ell=1}^{N}\left[\kappa_1 \mathcal{D}(a_\ell^+) + \kappa_2 \mathcal{D}(a_\ell^2)\right]}$$

with dissipative processes $\mathcal{D}(\hat{O}) := \hat{O}\rho\hat{O}^+ - \frac{1}{2}\{\rho, \hat{O}^+\hat{O}\}$

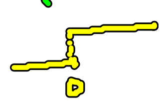
(new!)

nonlocal coupling of range d:



$$H = \hbar \sum_{\substack{m=1 \\ m \neq \ell}}^{N} K_{\ell m}\left(a_\ell^+ a_m + a_\ell a_m^+\right)$$

coupling matrix $K_{\ell m} = \frac{V}{2d}\,\Theta\left(d - |\ell - m|\right)$



coupling strength V

general form of Lindblad master eq.

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_\mu \gamma_\mu\left(L_\mu \rho L_\mu^+ - \frac{1}{2}\{\rho, L_\mu^+ L_\mu\}\right)$$

Lindblad op.

eff. Hamiltonian

$$H_{eff} := H - \frac{i\hbar}{2}\sum_p \gamma_p L_p^+ L_p$$

$$(3) \Rightarrow H_{eff} = \underbrace{i\hbar\kappa_1 \sum_{\ell=1}^{N} (a_\ell^+ a_\ell + 1)}_{\text{diss.}} - \underbrace{i\hbar\kappa_2 \sum_{\ell=1}^{N} \hat{n}_\ell(\hat{n}_\ell - 1)}_{\text{diss.}} \qquad \hat{n}_\ell := a_\ell^+ a_\ell$$

$$+\hbar \sum_{\substack{m=1\\ m\neq \ell}}^{N} K_{\ell m}(a_\ell^+ a_m + a_\ell a_m^+)$$

(nonlocal coupling)

Bose-Hubbard model with long-range interaction and complex on-site self-energies and chem. pot.

($\Rightarrow$ application to driven dissipation Bose-Einstein condensation)

Gaussian quantum fluctuations

expansion $b_\ell(t) := D^+(\alpha(t))a_\ell D(\alpha(t)) = \underbrace{\tilde{a}_\ell}_{\text{quantum fluct.}} + \underbrace{\alpha_\ell(t)}_{\text{mean-field}}$

with displacement op $D(\alpha) = e^{\alpha\tilde{a}^+ - \alpha^*\tilde{a}}$

$$\alpha(t) \equiv (\alpha_1, \alpha_2, \dots, \alpha_N)$$
$$\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N)$$

Semiclassical regime $|\alpha_\ell(t)| \gg \|\tilde{a}_\ell\|$

quantum fluct. gaussian

expansion as Eq.(1) $\Longrightarrow$ Master eq. in co-moving frame, describes quantum fluctuations

$$\Rightarrow (4) \boxed{\dot{\rho}_\alpha \simeq \underbrace{-\frac{i}{\hbar}[H_Q^{(\alpha)}, \rho_\alpha]}_{\text{coherent dyn.}} + 2\sum_{\ell=1}^{N}[\kappa_1 \mathcal{D}(\tilde{a}_\ell^+) + 4\kappa_2|\alpha_\ell|^2 \mathcal{D}(\tilde{a}_\ell)]}$$

$$\Downarrow$$

$$H_Q^{(\alpha)}(t) = \hbar\sum_{\ell=1}^{N}\Big(i\kappa_2(\alpha_\ell^*)^2\tilde{a}_\ell^2 - i\kappa_2\alpha_\ell^2(\tilde{a}_\ell^+)^2 + \sum_{n=1}^{N} K_{\ell\,\ell+n}(\tilde{a}_\ell^+\tilde{a}_{\ell+n} + \tilde{a}_\ell\tilde{a}_{\ell+n}^+)\Big)$$

$$+\hbar\sum_{\ell=1}^{N}\left(-i[\varepsilon_\ell\tilde{a}_\ell^+ - \varepsilon_\ell^*\tilde{a}_\ell] + i\kappa_1\alpha_\ell\tilde{a}_\ell^+ - i\kappa_1\alpha_\ell^*\tilde{a}_\ell\right)$$

$$+\hbar\sum_{\ell=1}^{N}\left(2i\kappa_2\alpha_\ell(\alpha_\ell^*)^2\tilde{a}_\ell - 2i\kappa_2\alpha_\ell^*\alpha_\ell^2\tilde{a}_\ell^+\right)$$

$$+\hbar\sum_{r=1}^{N} K_{\ell,\ell+r}\left(\alpha_{\ell+r}\tilde{a}_{\ell}^{+}+\alpha_{\ell+r}^{*}\tilde{a}_{\ell}+\alpha_{\ell}^{*}\tilde{a}_{\ell+r}+\alpha_{\ell}\tilde{a}_{\ell+r}^{+}\right)$$

semiclass. trajectory ($\Rightarrow$ lin. terms in Master eq. vanish):

$$(5)\quad \boxed{\dot{\alpha}_{\ell}(t)=\alpha_{\ell}(t)\left(\kappa_1-2\kappa_2|\alpha_{\ell}(t)|^2\right)-i\sum_{\substack{s=1\\ s\neq\ell}}^{N}K_{\ell s}\,\alpha_s(t)}$$

$$(4)\Rightarrow \frac{d}{dt}\langle\tilde{a}_i\rangle=\mathrm{tr}\left[\tilde{a}_i\,\dot{\varrho}_{\alpha}(t)\right]=\kappa_1\langle\tilde{a}_i\rangle-4\kappa_2|\alpha_i|^2\langle\tilde{a}_i\rangle$$

$$-2\kappa_2\alpha_i^2\langle\tilde{a}_i^{+}\rangle$$

$$-i\sum_{\substack{s=1\\ s\neq i}}^{N}K_{is}\langle\tilde{a}_s\rangle$$