Superradianz:

aber zuerst, Skizze der Quantisierung des Lichtfeldes, ausgehend von den Maxwell-Gl.:

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho, \quad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{B} = \frac{1}{c^2}\dot{\underline{E}} + \frac{1}{\varepsilon_0 c^2}\underline{j}, \quad \nabla \times \underline{E} = -\dot{\underline{B}}$$

(Cohen-Tannoudji, Dupont-Roc, Grynberg: Atom-Photon-Interactions, Appendix)

Fourier-Transformation: $\underline{E}(\underline{r},t) = \int d^3r\, e^{i\underline{k}\cdot\underline{r}}\, \underline{\mathcal{E}}(\underline{k},t)$

also:
$$i\underline{k}\cdot\underline{\mathcal{E}} = \frac{1}{\varepsilon_0}\rho(\underline{k},t)$$
$$i\underline{k}\cdot\underline{\mathcal{B}} = 0$$
$$i\underline{k}\times\underline{\mathcal{E}} = -\frac{\partial}{\partial t}\underline{\mathcal{B}}$$
$$i\underline{k}\times\underline{\mathcal{B}} = \frac{1}{c^2}\dot{\mathcal{E}} + \frac{1}{\varepsilon_0}\frac{1}{c^2}\underline{j}$$

$$\underline{k}\cdot\underline{\mathcal{E}} = \mathcal{E}_{\parallel} = \rho$$

$$\underline{k}\cdot\underline{\mathcal{B}} = 0 \quad \text{(rein transversal)}$$

$$\underline{\mathcal{B}} = \underline{\mathcal{B}}_{\perp}$$

Vektorpotential: $\underline{E}(\underline{r},t) = -\nabla\phi - \dot{\underline{A}}(\underline{r},t)$

$$\underline{B}(\underline{r},t) = \nabla\times\underline{A}$$

$$\underline{\mathcal{E}}(\underline{k},t) = i k\,\hat{\phi}(\underline{k},t) - \frac{\partial \hat{\underline{A}}}{\partial t}(\underline{k},t)$$

$$\underline{B}(\underline{k},t) = i\,\underline{k}\times\hat{\underline{A}}(\underline{k},t)$$

Verwende Eichfreiheit $\nabla\cdot\underline{A} = 0$

$A_\perp$ invariant (Coulombeichung)

Dynamik der Felder:

$$\underline{\alpha}(\underline{k},t) = \lambda(k)\left[\underline{\mathcal{E}}_\perp(\underline{k},t) + \hat{f}\,\underline{B}(\underline{k},t)\right]$$

$\hat{f}$ wirkt auf $\hat{B}$

$$\alpha(\underline{k},t) = \lambda(k)\left[\underline{\mathcal{E}}_\perp(\underline{k},t) - c\frac{\underline{k}}{|\underline{k}|}\,\underline{B}(\underline{k},t)\right]$$

$$\dot{\underline{\alpha}}(\underline{k},t) = \lambda(k)\left[\dot{\underline{\mathcal{E}}}_\perp - \frac{\underline{k}}{|\underline{k}|}\,\dot{\underline{B}}\right]$$

mit $\dot{\underline{B}} = -i\,\underline{k}\times\underline{\mathcal{E}}_\perp(\underline{k},t)$

$$\dot{\underline{\mathcal{E}}}_\perp = c^2 i\,\underline{k}\times\underline{B} - \frac{1}{\varepsilon_0}\hat{j}$$

$$\dot{\underline{\alpha}}(\underline{k},t) = \lambda(k)\left[-\frac{1}{\varepsilon_0}\hat{j} + c^2 i\,\underline{k}\times\underline{B} - \frac{c}{k}(-i)\left[k^2\,\underline{\mathcal{E}}_\parallel + \underline{k}\underbrace{(\underline{k}\cdot\underline{\mathcal{E}}_\perp)}_{=0}\right]\right]$$

$$= \lambda(k)\left[\frac{1}{\varepsilon_0}\hat{j} + \underbrace{i c k}_{=\omega}\left[c\frac{\underline{k}}{|k|}\times\underline{B} - \underline{\mathcal{E}}_\perp\right]\right]$$

es folgt $\dot{\underline{\alpha}}(\underline{k},t) = -i\omega\,\alpha(\underline{k},t) + \frac{i}{\sqrt{2\varepsilon_0\hbar\omega}}\hat{j}$

$$\lambda(k) = -i\sqrt{\frac{\varepsilon_0}{2\hbar\omega}}$$

(ohne Inhomogenität j, einfacher harmonischer Oszillation)

Lagrangedichte: $\mathcal{L} = \frac{\varepsilon_0}{2}\underline{E}^2 - \frac{1}{2\mu_0}\underline{B}^2$

$$\mathcal{L} = \frac{1}{2}\sum_{i=x,y,z}\left\{\varepsilon_0 \dot{A}_i^2 - \mu_0^{-1}(\nabla\times\underline{A})_i^2\right\}$$

einsetzen $\frac{\partial\mathcal{L}}{\partial f} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\frac{\partial f}{\partial t})} + \sum_j \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial(\frac{\partial f}{\partial x_j})}$

Coulomb-Eichung: $f_i = \{A_x, A_y, A_z\}$

Impulsvariable $\Pi_i = \Pi_{A_i} = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = \varepsilon_0\dot{A}_i$

Kanonische Quantisierung

$$\{A_i(\underline{r}), \Pi_j^*(\underline{r}')\}$$

$$\longrightarrow [\hat{A}_i(\underline{r}), \hat{\Pi}_j^+(\underline{r}')]_- = i\hbar\,\delta_{ij}\,\delta(\underline{r}-\underline{r}')$$

Diskutiere für Bosonen $[A,B]_- = AB - BA$

für Fermionen $[A,B]_+ = AB + BA$

Experimente: Bosonen zeigen Bose-Einstein-Kondensation $(-)$

Pauli-Prinzip für Fermionen $(+)$

$\underline{\alpha} \longrightarrow \hat{a}$, $[a_i^\dagger(\underline{k}), \hat{a}_i(\underline{k}')] = \delta_{ij}\,\delta_{\perp}(\underline{k}-\underline{k}')$

womit gilt:

$$H = \int d^3r\, \mathcal{H} = \int d^3r \left\{ \sum_i \left[\frac{\partial}{\partial \dot{x}} \hat{f}_i\right] \hat{\Pi}_i - \mathcal{L} \right\}$$

$$= \ldots = \sum_{\underline{k}}' \hbar\omega_{\underline{k}} \left(a_{\underline{k}}^\dagger a_{\underline{k}} + \frac{1}{2}\right)$$

Anwendung auf die spontane Emission von N-Atomen und dies im Operatorbild

$\longrightarrow$ Superradianz und Subradianz

$$H = \sum_{\alpha=1}^{N} \hbar\omega_\alpha\, \sigma_\alpha^+ \sigma_\alpha^- \quad \leftarrow \quad \sigma_\alpha^- \equiv |g\rangle_{\alpha\alpha}\langle e|$$

Flip-Operator

$\hbar\omega_\alpha \equiv$ Übergangsfrequenz

$$+ \sum_{\lambda=1,2} \int d^3k\, \hbar\omega_k\, b_{\lambda\underline{k}}^\dagger b_{\lambda\underline{k}}$$

λ = Polarisation

k = Wellenzahl

$k = \frac{2\pi}{\lambda}$

$$- \sum_{\alpha=1}^{N} \mu_\alpha \cdot \underline{\hat{E}}(\underline{x}_\alpha)\,(\sigma_\alpha^+ + \sigma_\alpha^-)$$

$\mu_\alpha \equiv$ reell

zerlegt das $\underline{E}$-Feld in $(+)$, $(-)$

$$\underline{E}(\underline{x}_\alpha) = \underline{E}^{(+)}(\underline{x}_\alpha) + \underline{E}^{(-)}(\underline{x}_\alpha)$$

$$\underline{E}^{(+)} = \sum_\lambda \int d^3k \; i \underbrace{\sqrt{\frac{\hbar\omega}{2\varepsilon_0(2\pi)^3}}}_{\text{Vakuum-Feldamplitude}} \underbrace{\underline{e}_{\lambda\underline{k}}}_{\text{Polarisationstensor}} b_{\lambda k} e^{i\underline{q}\cdot\underline{x}_\alpha}$$

$$\underline{E}^{(-)} = \left(\underline{E}^{(+)}\right)^+$$

Annahme: identische Atome $\omega_\alpha = \omega_{eg}$

ohne Überlapp $[\sigma_\alpha^{(\pm)}, \sigma_\beta^{(\pm)}] = 0$ $(\alpha \neq \beta)$

selbe Orientierung $\underline{\mu}_\alpha = \underline{\mu}$

$\rightarrow$ alle Atome koppeln an dasselbe Photonreservoir

Vorgehen: Herleitung der BWGl. eines beliebigen Operators a, über die Heisenberg BWGl.

(Photonreservoir ausgespart)

$$\dot{a} = \frac{i}{\hbar}[H, a] =$$

$$= i \sum_{\alpha=1}^{N} [\omega_{eg}\sigma_\alpha^+ \sigma_\alpha, a] +$$

$$+ \sum_\lambda \int d^3k \left[-\frac{i}{\hbar}\underline{\mu}\cdot\underline{\varepsilon}_{\lambda k} \left(\left[\sum_{\alpha=1}^{N} e^{i\underline{k}\cdot\underline{x}_\alpha} s_\alpha, a\right] \varepsilon_k b_{\lambda k} + \right.\right.$$

$$\left.\left. + \varepsilon_k^* b_{\lambda\underline{k}}^+ \left[\sum_{\alpha=1}^{N} e^{-i\underline{k}\cdot\underline{x}_\alpha} s_\alpha, a\right] \right) \right]$$

$$s_\alpha := \sigma_\alpha^+ + \sigma_\alpha^-$$

$$\dot{b}_{\lambda k} = \frac{i}{\hbar}[H, b_{\lambda k}] =$$

$$= -i\omega_k b_{\lambda k} + \frac{i}{\hbar} \mathcal{E}_k^* \underline{\mu} \cdot \underline{e}_{\lambda k} \left[\sum_{\alpha=1}^{N} e^{-i\underline{k}\cdot\underline{x}_\alpha} \sigma_\alpha\right]$$

Lösung über formale Integration

$$b_{\lambda k}(t) = b_{\lambda k}(0)\, e^{-i\omega_k t} +$$

$$+ \frac{i}{\hbar} \mathcal{E}_k^* \underline{\mu} \cdot \underline{e}_{\lambda k} \int_0^t dt'\, e^{-i\omega_k(t-t')} \sum_{\alpha=1}^{N} e^{-i\underline{k}\cdot\underline{x}_\alpha} \sigma_\alpha(t')$$

Einsetzen in die BWGl. für $a(t)$.
Wir sind sowieso nur an dem Vakuumfall interessiert

$$\dot{a}(t) =$$

$$= i \sum_{\alpha=1}^{N} \omega_{eg} [\sigma_\alpha^+ \sigma_\alpha^-, a]$$

$$+ \frac{1}{\hbar^2} \sum_\lambda \int d^3k\, \mathcal{E}_k |\underline{\mu} \cdot \underline{e}_{\lambda k}|^2 \int_0^t dt'\, e^{-i\omega_k(t-t')} \left[\sum_{\alpha,\beta} e^{i\underline{k}\cdot\underline{x}_{\alpha\beta}} [\sigma_\alpha^+(t) a(t)] \sigma_\beta(t')\right]$$

$$+ \text{h.a.} \qquad \underline{x}_{\alpha\beta} = \underline{x}_\alpha - \underline{x}_\beta$$

Auswerten $|\underline{\mu} \cdot \underline{e}_{\lambda k}|^2$, hierfür die Orthogonalität der Polarisationsrichtung verwenden [Scheck, Bd. 4, Quantisierung der Felder]

$$|\underline{\mu} \cdot \underline{e}_{\lambda k}|^2 = \sum_{\lambda\lambda'} \sum_{ij} \mu_i \mu_j\, e^i_{\lambda k} e^j_{\lambda' k}$$

$$= \sum_{ij} \mu_i \mu_j \underbrace{\sum_{\lambda,\lambda'} e^{i}_{\lambda k}\, e^{j}_{\lambda' k}}_{=\delta_{\lambda\lambda'}}$$

$$= \sum_{ij} \mu_i \mu_j \underbrace{\sum_{\lambda=1,2} e^{i}_{\lambda k}\, e^{j}_{\lambda k}}_{=\delta_{ij} - \frac{k_i k_j}{k^2}}$$

$$= \sum_i \mu_i^2 - \sum_{ij} \frac{k_i k_j \mu_i \mu_j}{k^2}$$

$$= \underline{\mu}^2 - (\hat{\underline{k}} \cdot \underline{\mu})^2 \qquad \hat{k} = \frac{\underline{k}}{|\underline{k}|}$$

$$= \mu^2 \left[1 - (\hat{k} \cdot \hat{\mu})^2\right]$$

$$\int d^3k \longrightarrow \int_0^\infty dk\, k^2 \underbrace{\int_0^{2\pi} d\varphi_k \int_0^{\pi} d\theta_k \sin\theta_k}_{\int d\Omega_k}$$

$$|\mathcal{E}_k| = \frac{\hbar\omega_k}{2\varepsilon_0 (2\pi)^3}$$

[Lehmberg PRA 2, 883 (1970)]

$$\dot{a}(t) = \frac{i}{\hbar}\left[\ldots, \ldots\right]$$

$$+ \frac{\mu^2}{2\hbar\varepsilon_0} \frac{c}{(2\pi)^3} \sum_{\alpha,\beta} \int d\Omega_k \left[1 - (\hat{\underline{k}} \cdot \hat{\underline{\mu}})\right] \circledast .$$

$$\circledast = \int_0^t dt' \int_0^\infty dk\, k^3\, e^{i\omega\left[t - t' - \frac{\hat{k} \cdot x_{\alpha\beta}}{c}\right]} \left[\sigma_\alpha(t), a(t)\right] \sigma_\beta(t')$$

$$+ \text{h.a.}$$