c) Die Diracsche „δ-Funktion"

· Grundeigenschaft: $\boxed{\int_{-\infty}^{\infty} \delta(x-x') f(x')\, dx' = f(x)}$ (5.46)

Satz:

$$\boxed{\begin{aligned} &\delta(x-x') = \text{"}\lim_{\varepsilon\to 0} \delta_\varepsilon(x-x')\text{"} \\ &\text{wobei } 1.\ \delta_\varepsilon(x-x') \underset{\xi=\frac{x-x'}{\varepsilon}}{=\!=} \frac{1}{\varepsilon} g(\xi) \\ &\qquad 2.\ \int_{-\infty}^{\infty} \delta_\varepsilon(x-x')\, \underbrace{d(x-x')}_{\varepsilon d\xi} = \int_{-\infty}^{\infty} g(\xi)\, d\xi = 1 \\ &\qquad 3.\ \text{"....": in (5.46) erst } \int \ldots, \text{ dann } \lim_{\varepsilon\to 0} \end{aligned}}$$ (5.47)

• Bsp.: (iv) $\delta_\varepsilon(x) = \frac{1}{\pi\varepsilon} \frac{\sin\frac{x}{\varepsilon}}{x/\varepsilon} = \frac{1}{2\pi} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} dq\, e^{\pm iqx}$ (5.51)

· Eigenschaften der δ-Funktion:

(i) $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \text{"}\infty\text{"}, & x = 0 \end{cases}$ (5.52)

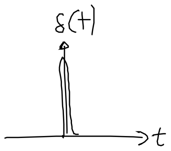
(ii) $\int_a^b \delta(x-x') f(x')\, dx' = \begin{cases} f(x), & a<x<b \\ 0, & \text{sonst} \end{cases}$ (5.53)

(iii) $\delta(ax) = \frac{1}{|a|} \delta(x)$ (5.54)

$\delta(-x) = \delta(x)$ .... gerade Fkt (5.55)

(iv) $\delta(x) \underset{(5.51)}{=\!=} \int_{-\infty}^{\infty} e^{\pm iqx} \frac{dq}{2\pi}$ (5.56)

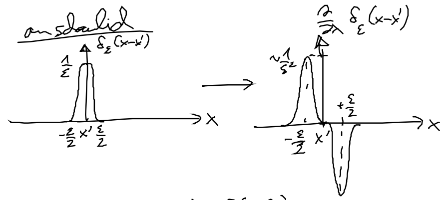
... Fourier-Darstellung



(v) $\int_{-\infty}^{\infty} \frac{\partial}{\partial x} \delta(x-x') f(x')\, dx' = -\int \left[\frac{\partial}{\partial x'} \delta(x-x')\right] f(x')\, dx' \overset{\text{Produktregel}}{=} f'(x)$

... Übungen

anschaulich

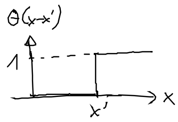


$\int \ldots \longrightarrow \frac{f(x+\frac{\varepsilon}{2}) - f(x-\frac{\varepsilon}{2})}{\varepsilon} \xrightarrow{\varepsilon\to 0} f'(x)$

Beweis: (iii)/(v) ... Übung

- Stufenfunktion:

(i) $\boxed{\Theta(x-x') = \begin{cases} 1, & x-x' > 0 \\ 0, & x-x' < 0 \end{cases}}$ (S.58)



(ii) $\Theta(x) \underset{S.53}{=} \int_{-\infty}^{x} \delta(x')\,dx'$ (S.59)

$\longrightarrow \boxed{\delta(x) = \frac{d}{dx}\Theta(x)}$ (S.60)

d) Der Formalismus der linearen Antwort II

- $x_{part}(t) = \int_{-\infty}^{\infty} G_c(t-t')\,F(t')\,dt'$ (S.42)

- Es galt: $\hat{L}(t)\,G_c(t-t') = \underbrace{\delta(t-t')}_{\delta\text{-Kraft}}$, $\hat{L}(t) = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2$ (S.61)

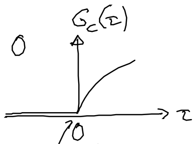
$\int \delta(t-t')dt' = 1$ ... δ-Kraftstoß

- Bestimmung der Greenschen Fkt. aus Stetigkeits- und Sprungbed.: $\tau = t - t'$

(i) Kausalität: $\boxed{G_c(\tau) = 0, \ \tau < 0}$

(ii) Stetigkeit: $G_c(\varepsilon) - \underbrace{G_c(-\varepsilon)}_{0} = 0$, $\varepsilon \to 0$

$G_c(0) = 0$



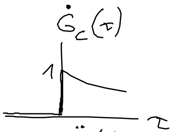
Knick [s. (iii)]

(iii) Sprungbeding.:

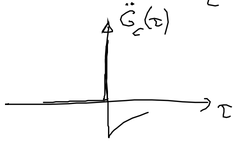
(S.61) $\int_{-\varepsilon}^{\varepsilon} d\tau \Big[\underbrace{\ddot{G}_c(\tau)} + \underbrace{2\gamma\,\dot{G}_c(\tau)} + \underbrace{\omega_0^2\,G_c(\tau)}_{\to 0,\ \varepsilon\to 0} - \underbrace{\delta(\tau)}_{=1}\Big] = 0$

$2\gamma\,[G_c(\varepsilon) - G_c(-\varepsilon)] \to 0, \ \varepsilon \to 0$ [s. (ii)]

$\dot{G}_c(\varepsilon) - \underbrace{\dot{G}_c(-\varepsilon)}_{=0\ (s.(i))}$



$\xrightarrow{\varepsilon\to 0} \boxed{\dot{G}_c(+0) = 1}$



vgl. δ-Kraftstoß: $1 = \int_{-\varepsilon}^{\varepsilon} \underbrace{\frac{F}{m}}_{\delta(\tau)}\,d\tau \underset{(2.1)}{\overset{F=\dot{p}}{=}} \frac{1}{m}\big[p(\varepsilon) - \underbrace{p(-\varepsilon)}_{0}\big]$

$\xrightarrow{\varepsilon\to 0} \boxed{1 = v(0) = \dot{x}(0)}$

(iv) $\tau > 0$: $G_c(\tau)$ löst homog. DGl.

Kausale Greensche Fkt.: (5.62)

$$\xrightarrow{\text{(i)-(iv)}} \boxed{G_c(t-t') = \underbrace{\Theta(t-t')}_{\text{(i)}}\ \underbrace{\frac{1}{\omega_d}}_{\text{(iii)}}\ \underbrace{e^{-\gamma(t-t')}\ \sin[\omega_d(t-t')]}_{\text{(ii),(iv): (5.20)},\ \varphi=\frac{\pi}{2}}}$$

• Bemerkungen:

(i) δ-Kraft: $\delta(t-t') \underset{(5.56)}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i\omega(t-t')} d\omega$ ... kontinuierliche Überlagerung von harm. Kräften der Stärke $\frac{1}{2\pi}$

$\longrightarrow$ Antwort

$\chi(\omega) \frac{1}{2\pi} e^{i\omega(t-t')}$ (*) vgl. (5.30)

Linearität der DGl:

$$\xrightarrow[\text{Überlagere (*)}]{\text{Antwort auf } \delta(t-t')} \boxed{G_c(t-t') = \int \chi(\omega) e^{i\omega(t-t')} \frac{d\omega}{2\pi} \equiv \chi(t-t')} \quad (5.63)$$

... Fourier-Integral-Darstellung von $G_c(t-t')$

dynamische Suszeptibilität $\chi(\omega) \equiv$ Fourier-Transformierte von $G_c(t-t')$

$$\boxed{G_c(\omega) = \chi(\omega)} \quad (5.64)$$

NB: Berechne (5.63) durch Integration im Komplexen [Mathe, später]

Einschub: Fouriertransformation (FT)

Vgl. Vektor: $\underline{a} = \sum_i a_i \underline{e}_i$ mit $a_i = \underline{e}_i \cdot \underline{a}$, $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

hier: Basis für $f(t)$?

definiere: Fouriertrafo von $f(t)$:

$$\hat{f}(\omega) \equiv f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

dann gilt: (o.B.)
$$f(t) = \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$$

Faltungssatz der FT:

$$f(\omega)\, g(\omega) \xrightarrow{\text{FT}} \int_{-\infty}^{\infty} f(t-t')\, g(t')\, dt'$$

(ii) allgemeine Kraft:

$$\hat{L}(t)\ \underbrace{x(t)}_{=\int x(\omega) e^{i\omega t} \frac{d\omega}{2\pi}} = \tilde{F}(t) = \int \hat{\tilde{F}}(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$$

$$\hat{L}(t)\, e^{i\omega t} = \chi^{-1}(\omega)\, e^{i\omega t} \;\Big|\; \int \chi^{-1}(\omega)\, x(\omega)\, e^{i\omega t} \frac{d\omega}{2\pi} = \int \tilde{F}(\omega)\, e^{i\omega t} \frac{d\omega}{2\pi}$$

$$\longrightarrow \boxed{\chi^{-1}(\omega)\, x(\omega) = \hat{F}(\omega)} \quad \text{... vgl. (5.30)}$$

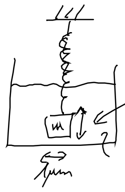
$$\longrightarrow x(\omega) = \chi(\omega)\, \hat{F}(\omega) \xrightarrow[\text{der FT}]{\text{Faltungssatz}}$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} \underbrace{\chi(t-t')}_{G_c(t-t')} \hat{F}(t')\, dt'}$$

vgl. (5.42)

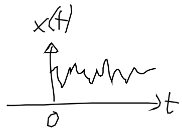
(iii) Ausblick: Fluktuations – Dissipations – Theorem

Experiment:



Zitterbewegung aufgrund der Stöße der Flüssigkeitsmoleküle

charakterisiere Zitterbewegung



$$C(t) = \langle x(t)\, x(0) \rangle_E \quad \text{.... zeitl. Autokorrelationsfkt. der Auslenkung } x(t)$$

Mittel über Ensemble

Führe ein: $C(\omega) = \int C(t)\, e^{-i\omega t}\, dt$

$$\xrightarrow{\text{o.B.}} \boxed{\underbrace{C(\omega)}_{\text{Fluktuationen}} = -\frac{2}{m} \frac{k_B T}{\omega} \underbrace{\chi''(\omega)}_{\text{Absorption/ Dissipation}}} \quad (5.65)$$

... Fluktuations – Dissipations – Theorem