

14.4. Liouville'sche Satz

Phasenraum dichte: $\rho(\underline{x}, t)$
 Wahrscheinlichkeit: $\rho(\underline{x}, t) d^{6N}X$ (14.31)

Normierung: $\int_V \rho(\underline{x}, t) d^{6N}X = 1$

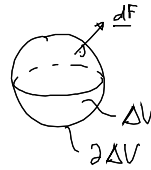
$$\underline{x}(t) = \{q_i(t), p_i(t)\} \quad (14.32)$$

Erwartungswert = Mittelwert einer Observable

$$\langle A \rangle = \int d^{6N}X \rho(\underline{x}, t) A(\underline{x}, t) \quad (14.33)$$

• Kontinuitätsgl.: Herleitung

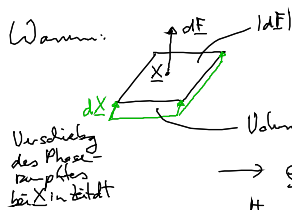
$$\begin{aligned} \frac{d}{dt} \int_{\Delta V} \rho(\underline{x}, t) d^{6N}X &= \int_{\text{fest } \Delta V} \frac{\partial}{\partial t} \rho(\underline{x}, t) d^{6N}X \\ &= - \int_{\partial \Delta V} \underline{j}(\underline{x}, t) \cdot d\underline{F} \quad (14.33) \end{aligned}$$



mit Wahrscheinlichkeitsstromdichte:

$$\underline{j}(\underline{x}, t) = \rho(\underline{x}, t) \dot{\underline{x}}(t) \quad (14.34)$$

Warum:



Volumen: Fläche $\times$ Höhe = $d\underline{x} \cdot d\underline{F}$

$\rightarrow \rho d\underline{x} \cdot d\underline{F} \dots$ Wahrscheinlichkeit, daß System in Zeit dt Fläche $|d\underline{F}|$ durchquert

$$\xrightarrow{\text{pro dt}} \rho \frac{d\underline{x}}{dt} \cdot d\underline{F} = \underline{j} \cdot d\underline{F}$$

Satz von Gauß
 (in 6N Dimensionen)

$$\begin{aligned} \int_{\partial \Delta V} \underline{j} \cdot d\underline{F} &= \int_{\Delta V} \text{div } \underline{j} d^{6N}X \\ \text{mit } \text{div} &= \nabla_{\underline{x}} \cdot = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{6N}} \end{pmatrix} \cdot \end{aligned} \quad (14.35)$$

$$(14.33) \xrightarrow{(14.34)} \int_{\Delta V} \left[\frac{\partial}{\partial t} \rho + \underbrace{\operatorname{div}(\rho \dot{X})}_{\neq 0} \right] d^3x = 0$$

$\frac{\Delta V}{\text{beliebig}} \rightarrow$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \dot{X}) = 0}$$

... Kontinuitätsgleichung für ρ

• Liouville'sche Satz:

Herleitung: $\operatorname{div}(\rho \dot{X}) = \sum_j \left[\frac{\partial}{\partial q_j} (\rho \dot{q}_j) + \frac{\partial}{\partial p_j} (\rho \dot{p}_j) \right]$

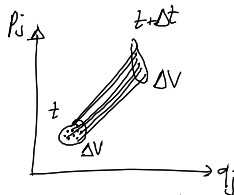
$$X = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{pmatrix} = \sum_j \left[\underbrace{\frac{\partial \rho}{\partial q_j}}_{=\frac{\partial H}{\partial p_j}} \dot{q}_j + \underbrace{\frac{\partial \rho}{\partial p_j}}_{=-\frac{\partial H}{\partial q_j}} \dot{p}_j + \rho \underbrace{\left(\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right)}_{=0} \right]$$

$$\operatorname{div}(\rho \dot{X}) \stackrel{(14.41)}{=} \{ \rho, H \} \quad (14.37)$$

$$(14.37) \text{ in } (14.36) \rightarrow \boxed{0 = \frac{\partial \rho}{\partial t} + \{ \rho, H \}} \quad (14.38)$$

... Liouville'scher Satz

(i) $(14.38) \xrightarrow{(14.13)} \frac{d\rho}{dt} = 0$... Dichte entlang Systembahn $X(t)$ ist konstant
 $\Rightarrow$ Ensemble von Systemen = "inkompressible Flüssigkeit"



(ii) thermodynam. Gleichgewicht:

$$\boxed{\frac{\partial \rho}{\partial t} = 0 \rightarrow \{ \rho, H \} = 0} \quad (14.39)$$

mögliche Lsg: $\rho = \rho(H) = \rho(H(q_j, \{p_j\})) \left[= \sum_n g_n H^n \right]$
 weil $\{H, H\} = 0$

Bsp: kanonische Ensemble:

$$\boxed{\rho = \frac{1}{Z} e^{-\frac{H}{k_B T}}}$$

$$Z = \int d^3x e^{-\frac{H}{k_B T}} \quad (14.40)$$

14.5 Kanonische Transformationen [lat. bestangepaßt]

• Motivierende Idee:

alle q_j zyklisch & $\frac{\partial H}{\partial t} = 0$

$$\rightarrow \dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \rightarrow p_j = \alpha_j = \text{konst.}$$

$$\rightarrow H = H(\alpha_1, \dots, \alpha_s) \rightarrow \dot{q}_j = \frac{\partial H}{\partial \alpha_j} = \omega_j(\alpha_1, \dots, \alpha_s) = \text{konst.}$$

$$\rightarrow q_j = \omega_j t + \beta_j \quad (14.41)$$

Bem: (i) (14.41) definiert integrable Systeme („die Ausnahme“...)
(ii) die Regel: nicht-integrable Systeme: Bsp: 3-Körper-Problem

Ziel: Trafos $\{q_j, p_j\} \rightarrow \{Q_k, P_k\}$, so daß alle Q_k zyklisch!

• Bisher: Punkttransformationen: $Q_k = Q_k(\{q_j\}, t)$ (14.42)

Beh: Punkttrafos lassen Lagrange Gln. forminvariant

Bew: Umhölz: $q_j = q_j(\{Q_k\}, t) \rightarrow \dot{q}_j = \sum_k \frac{\partial q_j}{\partial Q_k} \dot{Q}_k + \frac{\partial q_j}{\partial t} \quad (*)$

zeige: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$

$$= \sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{Q}_k} + \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \dot{Q}_k} \right) - \sum_j \left(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial Q_k} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial Q_k} \right)$$

(*) $\frac{\partial \dot{q}_j}{\partial \dot{Q}_k}$

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{Q}_k} + \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \dot{Q}_k} \right) - \left(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \dot{Q}_k} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{Q}_k} \right)$$

$$= \sum_j \underbrace{\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right)}_{=0} \frac{\partial q_j}{\partial \dot{Q}_k} = 0 \quad \checkmark$$

• jetzt:

Kanonische Trafos: $Q_k = Q_k(\{q_j\}, \{p_j\}, t)$

$P_k = P_k(\{q_j\}, \{p_j\}, t)$

wenn Hamiltonsche Kew. gl. forminvariant bleiben
bzgl. neuer Hamiltonfunktion $\bar{H} = \bar{H}(\{Q_k\}, \{P_k\}, t)$

äquivalent: $L(\{q_j\}, \{\dot{q}_j\}, t) \xrightarrow[\text{Trafo}]{\text{kanon.}} L(\{Q_k\}, \{\dot{Q}_k\}, t) + \frac{d}{dt} F \quad (14.44)$

Eid trafo
vgl. Kap. 13.5 c)

$\rightarrow$ Legendre Trafo:

$$\sum_j p_j \dot{q}_j - H = \sum_j P_j \dot{Q}_j - \bar{H} + \frac{d}{dt} F \quad (14.45)$$

• erzeugende Funktion F :

(i) bestimmt Trafo!

(ii) soll also von $\{q_j\}, \{p_j\}, \{Q_j\}, \{P_j\}$ abhängen
 $\underbrace{\hspace{10em}}_{\substack{2f \text{ unabhängig} \\ 2f \text{ abhängige Variable}}}$

→ 4 Möglichkeiten:

$$\begin{array}{ll} F_1(\{q_j\}, \{Q_j\}, t) & F_2(\{q_j\}, \{P_j\}, t) \\ F_3(\{p_j\}, \{Q_j\}, t) & F_4(\{p_j\}, \{P_j\}, t) \end{array} \quad (14.46)$$

(i) $F_1(\{q_j\}, \{Q_j\}, t)$:

Bilde: $\frac{d}{dt} F_1 = \sum_j \left(\frac{\partial F_1}{\partial q_j} \dot{q}_j + \frac{\partial F_1}{\partial Q_j} \dot{Q}_j \right) + \frac{\partial F_1}{\partial t}$

in (14.45) & Koeff. vgl. bei $\dot{q}_j, \dot{Q}_j$, da $\{q_j\}, \{Q_j\}$ unabh. Variable

$\frac{d}{dt} \dot{q}_j \rightarrow$	$p_j = \frac{\partial F_1}{\partial \dot{q}_j}$	Umkehr $\rightarrow$	$Q_k = Q_k(\{q_j\}, \{p_j\}, t)$
$\frac{d}{dt} \dot{Q}_j \rightarrow$	$P_j = -\frac{\partial F_1}{\partial \dot{Q}_j}$		$\rightarrow P_k = P_k(\{q_j\}, \{p_j\}, t)$
Rest $\rightarrow$	$H = H + \frac{\partial F_1}{\partial t}$		$= P_k(\{q_j\}, \{p_j\}, t)$

... F_1 erzeugt kanon. Trafo

(ii) $F_2(\{q_j\}, \{P_j\}, t) \xrightarrow[\text{Trafo}]{\text{Legendre}} F_1 + \sum_j P_j Q_j \quad (14.48) \quad \left[\text{vgl. Legendre Trafo} \right]$
 $p_j = -\frac{\partial F_1}{\partial \dot{q}_j}, \quad Q_j \rightarrow -P_j$

Thema der Legendre Trafo

$$\begin{array}{l} p_j = \frac{\partial F_2}{\partial \dot{q}_j} \\ Q_j = +\frac{\partial F_2}{\partial P_j} \\ H = H + \frac{\partial F_2}{\partial t} \end{array} \quad (14.49)$$

... F_2 erzeugt kanon. Trafo