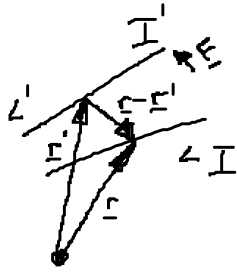


Nachtrag: Kraft zwischen 2 stromdurchflossenen Leitern



$$\underline{F} = - \frac{\mu_0}{4\pi} I I' \int_{L'} \int_L (d\underline{r} d\underline{r}') \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3}$$

Kraft von L' auf L

parallele Ströme
→ Anziehung
antiparallele Ströme
→ Abstoßung

2.4. Magnetische Multipole

Ausgangspunkt: $\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} d^3r' \frac{\underline{j}(\underline{r}')}{|\underline{r} - \underline{r}'|}$ Coulomb
Eichung
 $\text{div} \underline{A} = 0$

mit Randbed. $\underline{A}(\underline{r}) \rightarrow 0$ für $r \rightarrow \infty$



$$r' \ll r$$

$$\left[\frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\xi) \right]$$

$$\frac{1}{|\underline{r} - \underline{r}'|} = \frac{1}{r} + \frac{(\underline{r} \cdot \underline{r}')}{r^3} + \dots$$

$$\Rightarrow \underline{A}(\underline{r}) = \underbrace{\frac{\mu_0}{4\pi r} \int_{\mathbb{R}^3} d^3r' \underline{j}(\underline{r}')}_{\textcircled{1}} + \underbrace{\frac{\mu_0}{4\pi r^3} \int_{\mathbb{R}^3} d^3r' (\underline{r} \cdot \underline{r}') \underline{j}(\underline{r}')}_{\textcircled{2}} + \dots$$

Monopol-Term:

$$\text{mit } \nabla_{r'} \cdot (x_k' \underline{j}(r')) = x_k' \underbrace{(\nabla_{r'} \cdot \underline{j})}_0 + \underbrace{\underline{j} \cdot (\nabla_{r'} x_k')}_{\underline{j} \cdot \delta_{kk}} = j_k$$

$$\text{folgt: } \int_{\mathbb{R}^3} d^3 r' j_k(r') = \int_{S(\infty)} d\underline{f} \cdot x_k' \underline{j} = \underline{0}$$

Monopol-Term
verschwindet!

Dipol-Term:

$$[b(ac) - c(ab)]$$

$$\begin{aligned} \text{Mit } (\underline{r}' \times \underline{j}) \times \underline{r} &= (\underline{r} \cdot \underline{r}') \underline{j} - (\underline{r} \cdot \underline{j}) \cdot \underline{r}' \\ &= \underline{2}(\underline{r} \underline{r}') \underline{j} - [(\underline{r} \underline{r}') \underline{j} + (\underline{r} \cdot \underline{j}) \underline{r}'] \end{aligned}$$

$$\text{und } \nabla_{r'} \cdot \left\{ x_k' (\underline{r} \cdot \underline{r}') \underline{j} \right\} = \left[(\underline{r} \cdot \underline{r}') j_k + x_k' (\underline{r} \cdot \underline{j}) + x_k' (\underline{r} \cdot \underline{r}') \underbrace{\nabla_{r'} \cdot \underline{j}}_0 \right]$$

$$\begin{aligned} \text{folgt } \int_{\mathbb{R}^3} d^3 r' \nabla_{r'} \cdot \left\{ x_k' (\underline{r} \cdot \underline{r}') \underline{j} \right\} &= \int d\underline{f} \left[(\underline{r} \underline{r}') j_k + (\underline{r} \cdot \underline{j}) x_k' \right] \stackrel{!}{=} 0 \\ &\underbrace{\int_{S_\infty} d\underline{f} \left\{ x_k' (\underline{r} \underline{r}') \underline{j} \right\}}_{\substack{0 \text{ da } \underline{j} \xrightarrow{r' \rightarrow \infty} 0}} \end{aligned}$$

Also [...] gibt keinen Beitrag

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi r^3} \frac{1}{2} \int_{\mathbb{R}^3} d^3r' (\underline{r}' \times \underline{j}(\underline{r}')) \times \underline{r} \quad \underline{\text{Dipolterm}}$$

magnetisches Dipolmoment

$$\underline{m} := \frac{1}{2} \int d^3r' \underline{r}' \times \underline{j}(\underline{r}')$$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi r^3} \underline{m} \times \underline{r}$$

Elektrostatik

$$\text{analog } \phi(\underline{r}) = \frac{1}{4\pi\epsilon_0 r^3} \underline{p} \cdot \underline{r}$$

$$\underline{p} := \int_{\mathbb{R}^3} d^3r' \underline{r}' \rho(\underline{r}')$$

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^5} [3(\underline{p} \cdot \underline{r}) \underline{r} - r^2 \underline{p}]$$

Magnet. Induktion des Dipolmomentes \underline{m} :

$$\underline{B}(\underline{r}) = \text{rot} \left[\frac{\mu_0}{4\pi r^3} \underline{m} \times \underline{r} \right] = \frac{\mu_0}{4\pi} \frac{1}{r^5} [3(\underline{m} \cdot \underline{r}) \underline{r} - r^2 \underline{m}]$$

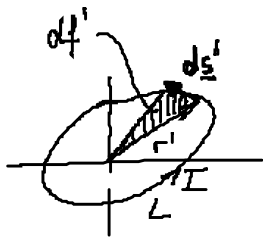
$$\text{wegen: } \text{rot}(\underline{v}_1 \times \underline{v}_2) = (\underline{v}_2 \cdot \nabla) \underline{v}_1 - (\underline{v}_1 \cdot \nabla) \underline{v}_2 + \underline{v}_1 (\nabla \cdot \underline{v}_2) - \underline{v}_2 (\nabla \cdot \underline{v}_1)$$

$$\underline{v}_1 = \frac{\underline{m}}{r^3}, \quad \text{div } \underline{v}_1 = -\frac{3 \underline{m} \cdot \underline{r}}{r^5}$$

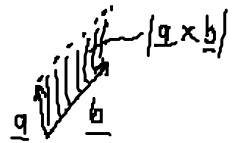
$$\underline{v}_2 = \underline{r}, \quad \text{div } \underline{v}_2 = 3$$

$$(\underline{v}_2 \cdot \nabla) \underline{v}_1 = -3 \underline{m} \frac{r^2}{r^5}, \quad (\underline{v}_1 \cdot \nabla) \underline{v}_2 = \frac{\underline{m}}{r^3}$$

Beispiel: (i) ebene Leiterschleife L



$$d\Omega' = \frac{1}{2} \underline{r}' \times d\underline{s}' \quad (*)$$



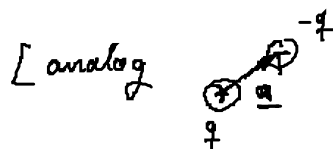
$$d^3 \underline{j}(\underline{r}') = d\underline{s}' \underline{I}$$

↑ Strom durch L

$$\Rightarrow \underline{m} = \frac{1}{2} \oint_L d^3 \underline{r}' \underline{r}' \times \underline{j}(\underline{r}') = \frac{I}{2} \oint_L \underline{r}' \times d\underline{s}' = I \int_F d\underline{\Omega}' = \underline{I} \cdot \underline{F} \cdot \underline{n}$$

\underline{n} : Normale auf der von L eingeschl. Fläche

also Ringstrom \rightarrow magn. Dipolmoment \underline{m}



$$\underline{p} = q \cdot \underline{a}$$

(ii) Bewegte Ladungen

N Teilchen mit Masse m_i und Ladungen q_i

spezifische Ladung $\frac{q_i}{m_i} = \frac{q}{m} = \text{const.}$

$$\rho(\underline{r}) = \sum_i q_i \delta(\underline{r} - \underline{r}_i)$$

$$\underline{j}(\underline{r}) = \sum_i q_i \underline{v}_i \delta(\underline{r} - \underline{r}_i) \quad \text{mit Geschwindigkeit } \underline{v}_i = \frac{d\underline{r}_i}{dt}$$

magn. Dipolmoment:

$$\underline{m} = \frac{1}{2} \int d^3 \underline{r}' \underline{r}' \times \underline{j}(\underline{r}') = \frac{1}{2} \sum_i q_i \int d^3 \underline{r}' \underline{r}' \times \underline{v}_i \delta(\underline{r}' - \underline{r}_i)$$

$$\begin{aligned}
&= \frac{1}{2} \sum_i q_i (\underline{r}_i \times \underline{v}_i) = \frac{1}{2} \sum_i \underbrace{\frac{q_i}{m_i}}_{q/m} m_i (\underline{r}_i \times \underline{v}_i) \\
&= \frac{q}{2m} \underbrace{\sum_i m_i (\underline{r}_i \times \underline{v}_i)}_{\underline{L}} = \frac{q}{2m} \underline{L} \\
&\qquad\qquad\qquad \underline{L} \quad \underline{L} \text{ Bahndrehimpuls}
\end{aligned}$$

$$\underline{m} = \frac{q}{2m} \underline{L}$$

gilt auch für starre Körper

(aber nicht für Spin eines Elektrons

$$\underline{m} = g \frac{e}{2m} \underline{S} \quad \text{mit } g \approx 2 \quad \text{siehe QM}$$

Kraft auf Stromverteilung $\underline{j}(\underline{r}') = \rho(\underline{r}') \underline{v}(\underline{r}')$

im Feld einer magn. Induktion $\underline{B}(\underline{r}')$ (extern):

$$\underline{F} = \int_{\mathbb{R}^3} d^3r' \underline{j}(\underline{r}') \times \underline{B}(\underline{r}') \quad \text{Lorentz Kraft}$$

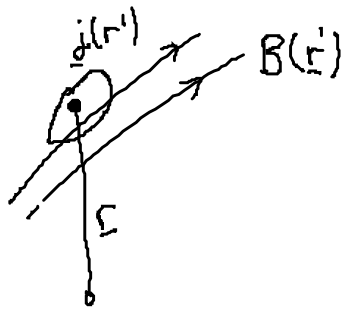
Taylorentwicklung: (Probodipol im geg. \underline{B})

$$\underline{B}(\underline{r}') = \underline{B}(\underline{r}) + [(\underline{r}' - \underline{r}) \cdot \nabla_{\underline{r}}] \underline{B}(\underline{r}) + \dots$$

$$\Rightarrow \underline{F} = \underbrace{\left[\int d^3r' \underline{j}(\underline{r}') \right]}_{\text{keine Monopole } \circ} \times \underline{B}(\underline{r}) + \int d^3r' \underline{j}(\underline{r}') \times [(\underline{r}' - \underline{r}) \cdot \nabla_{\underline{r}}] \underline{B}(\underline{r}) + \dots$$

$$= \int d^3 r' \underline{j}(\underline{r}') \times \underbrace{(\underline{r}' \cdot \nabla_r) \underline{B}(\underline{r})}_{\nabla_r (\underline{r}' \cdot \underline{B}(\underline{r}))} - \underbrace{\int d^3 r' \underline{j}(\underline{r}') \times (\underline{r}' \cdot \nabla_r) \underline{B}(\underline{r})}_{\underline{0}} + \dots$$

(externes Feld soll keine Stromwirbel im Bereich $\underline{j}(\underline{r}')$ haben)



$$= \int d^3 r' \underline{j}(\underline{r}') \times \nabla_r (\underline{r}' \cdot \underline{B}(\underline{r})) - \nabla_r \times [(\underline{r}' \cdot \underline{B}) \underline{j}(\underline{r}')] + \underbrace{(\underline{r}' \cdot \underline{B}) (\nabla_r \times \underline{j}(\underline{r}'))}_{\underline{0}}$$

$$= - \nabla_r \times (\underline{m} \times \underline{B}(\underline{r}))$$

$$= (\underline{m} \cdot \nabla_r) \underline{B}(\underline{r}) = - \nabla_r (- \underline{m} \cdot \underline{B}(\underline{r}))$$

$$\Rightarrow \boxed{V = - \underline{m} \cdot \underline{B}(\underline{r})}$$

Potenzielle Energie eines Dipols im Magnetfeld.

[analog : $V = - \underline{p} \cdot \underline{E}(\underline{r})$ für elektrischer Dipol im el. Feld]