We consider Langevin equations:

(mechanical system)

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\gamma(x,v) - \dfrac{\partial U(x)}{\partial x} + \sqrt{2D}\,\xi(t) \end{cases}$$

two-dimensional oscillatory system driven by additive noise.

x and v are the position and the velocity of a particle which moves in the potential $U(x)$ with friction $\gamma(x,v)$. If $\gamma = \text{const}$ we deal with damped oscillations with a nonlinear potential. (conservative)

If the damping term is nonlinear we have a dissipative nonlinear system with U being a harmonic potential in most cases.

The FPE for the transition probability density $p_2 = p_2(x, v, t \,/\, x_0, v_0, t_0)$:

$$\frac{\partial}{\partial} p_2 + v\frac{\partial}{\partial x} p_2 - \frac{\partial U}{\partial x}\frac{\partial}{\partial v} p_2 = \frac{\partial}{\partial v}\gamma(x,v)v\cdot p_2 + D\frac{\partial^2}{\partial v^2} p_2$$

In the stationary limit p_2 tends to its stationary density, $p_2 \to p^s(x,v)$ with $\frac{\partial p^s}{\partial t} = 0$ and does not depend on the initial state

Amplitude and phase description

Lecture 14
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An analytical approach for the case of harmonic potential and weak dissipative forces. The main idea:
We search for solutions of the deterministic problem by transforming to an amplitude and phase description.
Small dissipation => one may assume a slow variation of the amplitude of oscillations A and a shift φ of the phase $\Phi = \Phi_0 + \varphi$ added to the fast motion $\Phi_0 = \omega_0 t$, where ω_0 is the natural frequency of the oscillator.

Let us consider the harmonic potential $U(x) = \omega_0^2 x^2/2$ and the following transformation of variables:

$$x = A\cos(\omega_0 t + \varphi), \quad v = -\omega_0 A \sin(\omega_0 t + \varphi)$$

The Langevin equations for the amplitude A and the phase shift φ are

$$\dot{A} = \frac{\sin(\omega_0 t + \varphi)}{\omega_0} \gamma(x, v) v + y_A,$$
$$\dot{\varphi} = \frac{\cos(\omega_0 t + \varphi)}{A\omega_0} \gamma(x, v) v + y_\varphi,$$

where $y_A(t)$ and $y_\varphi(t)$ are new noise sources:

$$y_A = -\frac{\sin(\omega_0 t + \varphi)}{\omega_0}\sqrt{2D}\,\xi(t), \quad y_\varphi = -\frac{\cos(\omega_0 t + \varphi)}{A\omega_0}\sqrt{2D}\,\xi(t)$$

The first term in both equations for the amplitude and phase can be treated by averaging over one period $T = 2\pi/\omega_0$ under the assumption of constant A and φ. The noise in the equations for A and φ is multiplicative.

This procedure gives the first-order expansions $f_A(a, \varphi)$ and $f_\varphi(a, \varphi)$ of the considered theory.

In the first kinetic coefficient one obtains from the noise [Eq(51) Lecture 9]

$$K_1^A \sim D \frac{\cos^2(\omega_0 t + \varphi)}{A\omega_0^2}, \quad K_1^\varphi \sim -D \frac{\cos(\omega_0 t + \varphi)\sin(\omega_0 t + \varphi)}{A^2\omega_0^2}$$

After averaging over one period only one nonvanishing contribution $\frac{D}{2A\omega_0^2}$ remains in K_1^A. The second (kinetic) moments can be calculated following Eq. (52) [see Lecture 9] and after averaging over one period the cross-correlations vanish $\Rightarrow$

$\Rightarrow K_2^{A,\varphi} = 0$. The noise sources of the amplitude and the phase shift are, therefore, independent Gaussian white noises with the intensities:

$$K_2^{A,A} = \frac{D}{2\omega_0^2}, \quad K_2^{\varphi,\varphi} = \frac{D}{2\omega_0^2 A^2}.$$

Thus, the FPE in the amplitude-phase variables for the transition probability density $p_2(A, \varphi, t \mid A_0, \varphi_0, t_0)$:

$$\frac{\partial}{\partial t} p_2 = -\frac{\partial}{\partial A}\left(f_A + \frac{D}{2 A \omega_0^2}\right) p_2 - \frac{\partial}{\partial \varphi} f_\varphi p_2 +$$

$$+ \frac{D}{2\omega_0^2} \frac{\partial^2 p_2}{\partial A^2} + \frac{D}{2\omega_0^2 A^2} \frac{\partial^2 p_2}{\partial \varphi^2}$$

The Rayleigh distribution solves in the stationary case this equation with $\gamma = \text{const}$ and a harmonic potential with frequency ω_0

$$P^s(A, \varphi) = \frac{1}{2\pi} \frac{A}{\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right),$$

$$\sigma^2 = \frac{D}{\omega_0^2 \gamma}.$$

This approach can be applied to many stochastic oscillators [Kuznetsov, Stratonovich, Tikhonov, Nonlinear Transformation of Stochastic Processes (1965); Hänggi, Riseborough, Am. J. Phys. 51, 347 (1883); Shidlovskaya, Schimansky-Geier, Romanovski, Z. Phys. Chemie 214, 65 (2000); Stratonovich, Landa In: Nonlinear Transformation of Stochastic Processes, (1965), p. 259]

Energy Representation

An important representation of FPE uses energy as a variable. This representation is preferable for the cases of slow energy variations.

Let the friction be linear: $\gamma = \text{const}$. Then the velocity can be replaced by the energy as:

$$v(x, E) = \pm\sqrt{2[E - U(x)]},$$

using the definition of mechanical energy. As a result we obtain Langevin equations (after substituting v into the eq-s):

$$\dot{x} = v(x, E), \quad \dot{E} = -\gamma v^2(x, E) + \sqrt{2D}\, v(x, E)\, \xi(t)$$

The equation for the energy contains multiplicative noise

$y_{E,x} = \sqrt{2D}\, v(x, E)\, \xi(t) \rightarrow$ source of fluctuations in this representation.

Kinetic coefficients can be found:

$$K_1^x = v(x, E), \quad K_1^E = -\gamma v^2(x, E) + D, \quad K_2^{E,E} = D v^2(x, E)$$

Thus, the FPE in a coordinate and energy representation for the transition probability density $p_2 = p_2(x, E, t \,|\, x_0, E_0, t_0)$ is

$$\frac{\partial p_2}{\partial t} = -\frac{\partial}{\partial x} v(x, E) p_2 + \frac{\partial}{\partial E}\left(\gamma v^2(x, E) - D\right) p_2 + D \frac{\partial^2}{\partial E^2} v^2(x, E) p_2$$

The stationary solution:

$$p^s(x, E) = \frac{N}{|v(x, E)|} \exp\left(\frac{-\gamma E}{D}\right).$$

Integration over x in regions where $E \geqslant U(x)$ removes the dependence of the stationary probability density on the coordinate:

$$p^s(E) \sim T(E) \exp\left(\frac{-\gamma E}{D}\right)$$

with $T(E)$ being the period of a closed trajectory for a given energy E. In particular, for a Hamiltonian function containing d quadratic terms, the stationary probability density is

$$p^s(E) \sim E^{\frac{d}{2}-1} \exp\left(-\frac{\gamma E}{D}\right)$$