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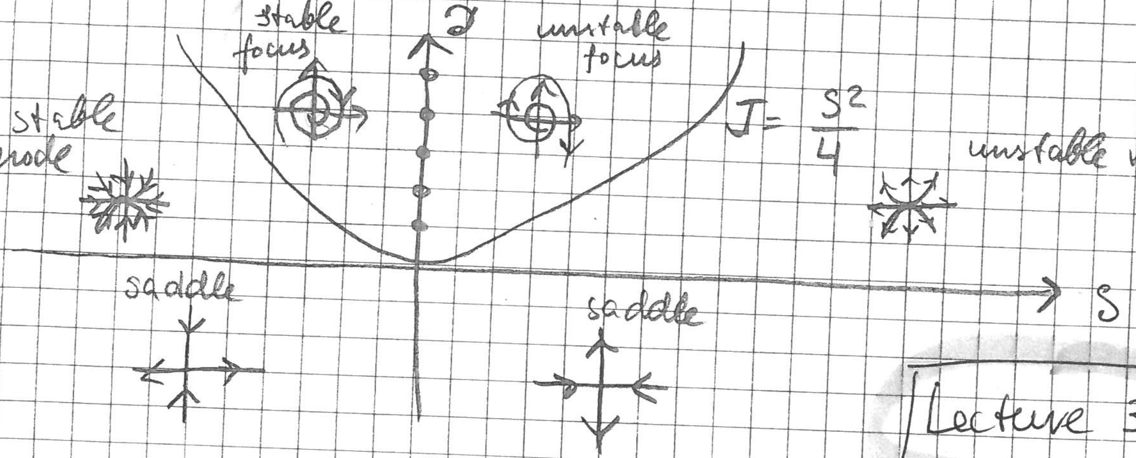

Lecture 3
02.11.2016

$S = 0 \Rightarrow$ center

Conservative systems: they have two types of steady states:
center, saddle

Dissipative: stable/unstable node, focus; saddle

Attractors: stable node and focus

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van der Pol oscillator

$$\ddot{x} + \underbrace{\mu(x^2-1)}\dot{x} + \overset{\omega^2}{x} = 0$$

$\mu \geq 0$ is a parameter

This equation arose in connection with the nonlinear electrical circuits used in the first radios.

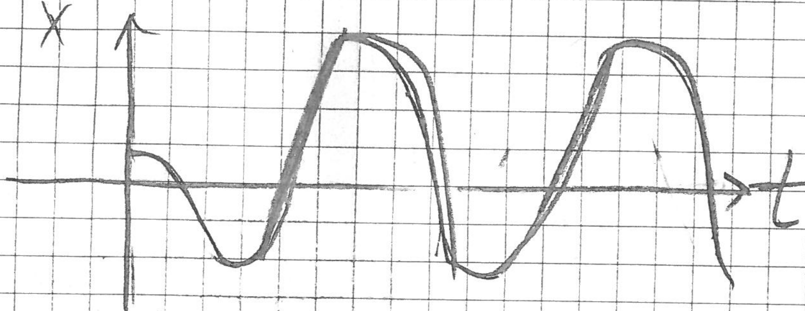
Balthasar van der Pol
Dutch physicist
1989 - 1959

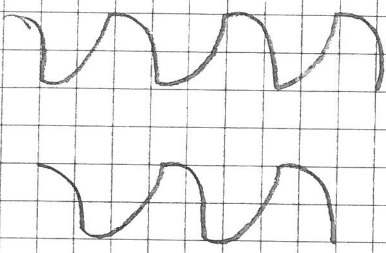
electrical circuits

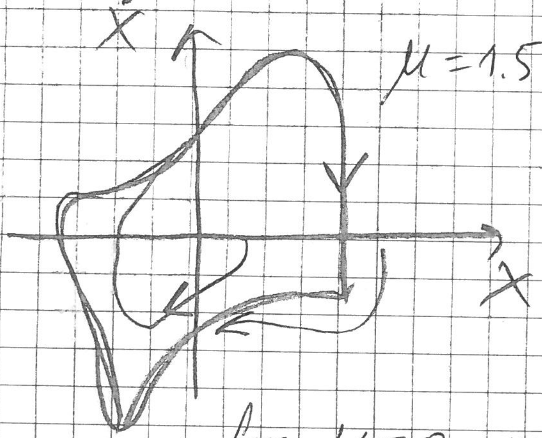
important model used in the first radio devices

Harmonic oscillator with nonlinear damping term
$\mu(x^2-1)\dot{x}$. This term acts as a positive (ordinary/usual) damping for $|x| > 1$, but as negative damping for $|x| < 1$.

$\Rightarrow$ In other words it causes large-amplitude oscillations to decay and (but), it pumps them back up if they become too small.







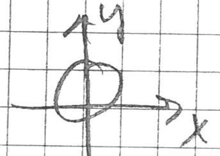
Self-sustined oscillation: the energy dissipated over one cycle balances the energy pumped in.

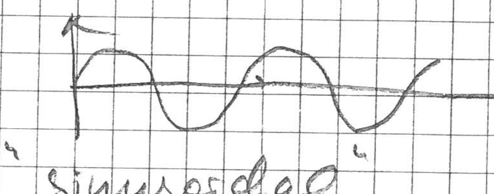
for $\mu > 0$ we have limit cycle

If the oscillations (amplitude) become too large ⇒ ⇒ the ~~nonlinear~~ damping is responsible for making them smaller

if — " — too small ⇒ — " — making them larger

S-L oscillator

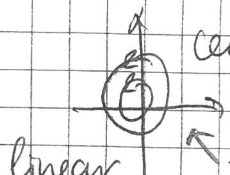




"sinusoidal"

There is no dependence on IC

- dissipative
- nolinear → in such systems LC are possible!



Not a limit cycle! Closed orbit, but not ISOLATED

Linear systems can have closed orbits, but they are not isolated!

- the oscillations will depend on IC!

3

Bifurcations — qualitative change of the phase portrait (caused by the parameter tuning) which takes place when one or several control parameters are varied.

bifurcation value of the parameter

- profound structural changes of the phase portrait
- appearance / disappearance of limit sets
- changes in the stability of trajectories

Hopf bifurcation - example

limit cycle birth / death bifurcation

=

Hopf

supercritical (soft)

subcritical (hard)

appearance / disappearance of a limit cycle

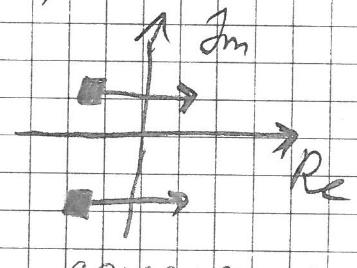
Hopf bifurcation

In DS with $N \geq 2$

there is a situation when a pair of complex conjugate eigenvalues of an equilibrium of stable focus type intersects the imaginary axis. ⇒

$\Rightarrow \operatorname{Re} \lambda_{1,2} = 0.$

$\operatorname{Im} \lambda_{1,2} \neq 0$



appearance

Lev Landau → Russian

$\boxed{\text{Stuart-Landau oscillator}}$ → Hopf normal form

means that it is the simplest model describing this bifurcation

$$\dot{z} = (\lambda + i\omega - |z|^2) z \; ; \quad z = x + iy = re^{i\varphi}$$

$\lambda = 0 \rightarrow$ Hopf bif. (super)

$$\dot{z} = (\lambda + i\omega + |z|^2 - |z|^4) z \text{ (subcritical)}$$

One can write one general equation

$$\dot{z}(t) = (\lambda + i\omega_0 - a|z(t)|^2 - b|z(t)|^4 z(t))$$

$a = 1,\ b = 0$ "super"

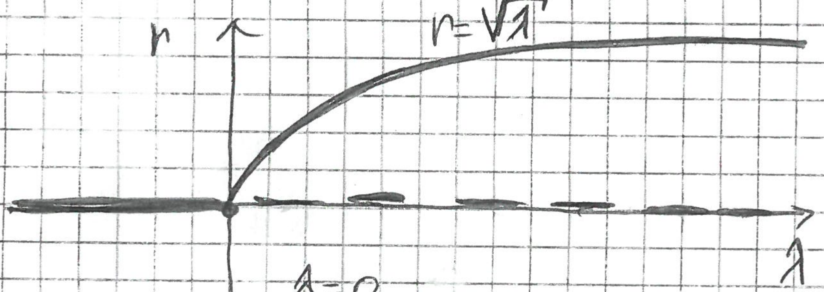
$a = -1,\ b = 1$ "sub"

$z = x + iy = re^{i\varphi}$

Super (soft)

Decompose z into real and imag. parts:

$$\begin{cases} \dot{r} = \lambda r - ar^3 - br^5 \\ \dot{\varphi} = \omega_0 \end{cases}$$



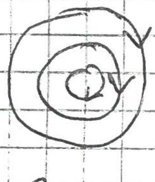
phase-parametric diagram

$\lambda = 0$
Hopf (super)

$\lambda < 0$ $\boxed{x = 0,\ y = 0}$ stable focus

stable focus

eigenvalues complex (negative real parts)



$\lambda = 0$ center

eigenvalues (pure) imaginary

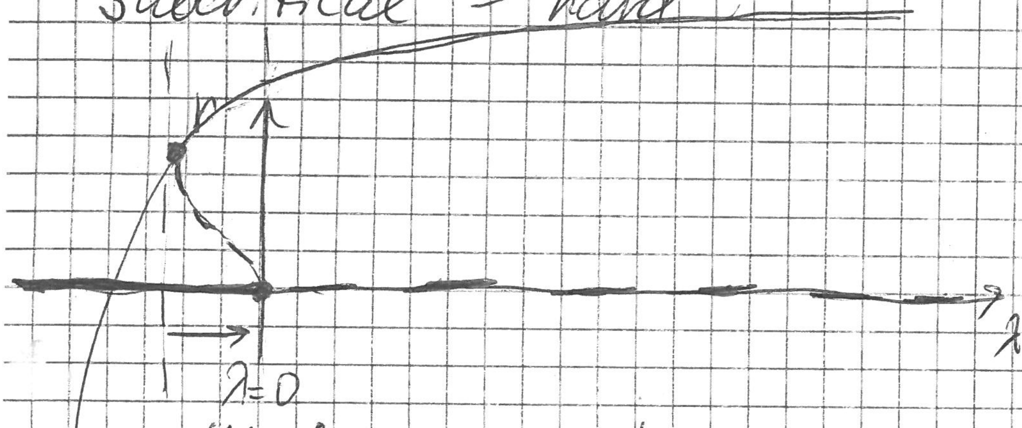
$\lambda > 0$

self-sustained osc. with freq. ω_0

limit cycle

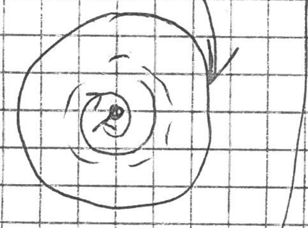
eigenvalues of the st. state are complex with positive real parts

Subcritical - "hard"



(Hopf subcritical)

saddle-node bifurcation of limit cycles



→ If we increase λ within the bistability domain → the stable limit

bistability: coexistence of 2 stable states: limit cycle + focus

cycle is growing the unstable gets smaller and smaller until it meets the stable focus and makes it unstable / unstabe limit cycle disappears

Attractor is difficult to define in a rigorous way

a set to which all neighboring trajectories converge.

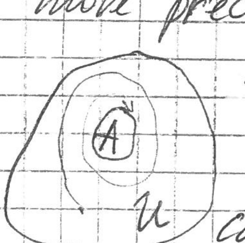
- stable fixed points
- stable limit cycles

In the determ. case we can define "attractor" more precisely p 331 (Strogatz)

Attractor is a closed set A with the fol. properties:

1) A is an invariant set: any trajectory $x(t)$ that starts in A stays in A for all time

2) A attracts an open set of IC: there is an open set $\mathcal{U}$ containing A such that if $x(0) \in \mathcal{U}$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all traj. that start sufficiently close to it. $\mathcal{U}$ is called basin of attraction of A



3) A is minimal: there is no proper subset of A that satisfies conditions 1 and 2.