Derivatives and Integrals

of a stochastic process $x(t)$.

Lecture 8
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$$\dot{x}(t) = \frac{dx(t)}{dt} = \lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t)}{\varepsilon}$$

The existence of the limit can be understood differently as well as the existence of the derivative. We will restrict ourselves in the definition of the limit to the mean square:

$$\lim_{\varepsilon \to 0} \left\langle \left| \frac{x(t+\varepsilon) - x(t)}{\varepsilon} - \dot{x}(t) \right|^2 \right\rangle = 0$$

Integrals are also defined in the mean square limit.

Let T_n be a partition/segmentation of the interval $T = [t_0, t]$ which becomes dense in the limit $n \to \infty$ in T. Then the integral over a stochastic process multiplied by a time-dependent deterministic function is also a stoch. proc.:

$$\mathcal{J}^f_\xi(t, t_0) = \int_{t_0}^{t} ds\, f(s)\, x_\xi(s) = \lim_{n \to \infty} q.m. \sum_{k=0}^{n-1} f(t_{k,n})\, x_\xi(t_{k,n}) \cdot (t_{k+1}^{(n)} - t_k^{(n)}), \text{ where } t_0^{(n)} = t_0,$$

$t_n^{(n)} = t$ and $t_k^{(n)} \leq t_{k,n} \leq t_{k+1}^{(n)}$. The notation q.m. stands for the mean square limit. If x_ξ is piecewise continuous the partition/segmentation does not play a role. For the existence of the integral the convergence of the following is sufficient and necessary:

$$\langle (\mathcal{J}^f_\xi(t, t_0))^2 \rangle = \int_{t_0}^{t} \int_{t_0}^{t} ds_1\, ds_2\, f(s_1) f(s_2) \langle x_\xi(s_1)\, x_\xi(s_2) \rangle$$

A more rigorous approach replaces $x_\xi(t_{k,n})(t_{k+1}^{(n)} - t_k^{(n)})$ with the increments $\Delta \mathcal{J}_{t_k} = \mathcal{J}^{f=1}_\xi(t_{k+1}, t_k)$ which can be considered as stochastic processes with certain properties. Such a formulation allows the treatment of stochastic integrals even if the increments are independent and, hence, $\mathcal{I}_\xi^{f=1}(t,t_0)$ is nondifferentiable. The convergence of a stochastic integral with smooth $f(t)$ for the case $t_{k,n}=t_k^{(n)}$ was proven by Ito:

$$\mathcal{I}_\xi(t,t_0)=\int_{t_0}^{t} d\mathcal{I}_s\, f(s)=\lim_{n\to\infty} q.m. \sum_{k=0}^{n-1} \Delta \mathcal{I}_{t_k^{(n)}}\, f(t_k^{(n)}) \quad (*)$$

further specification is necessary if f is random and a function of ξ, as in the case of multipl. noise

Ito integral

Let $x(t)=[x_1(t),\dots,x_n(t)]$ be a vector of state variables of a dynamical system. In the most general form the Langevin equation can be written as follows:

$$\boxed{\dot{x}_i(t)=f_i(x_1,\dots,x_n,t)+g_i(x_1,\dots x_n,\xi_1(t)\dots\xi_m(t),t)}$$

deterministic part

$f(x,t)=[f_1(x,t),\dots,f_n(x,t)]$

stochastic part

$g(x,\xi,t)=[g_1(x,\xi,t),\dots g_n(x,\xi,t)]$

with $g_i(x,\xi=0,t)=0$.

The multidimensional stoch. process $\xi(t)=[\xi_1(t),\dots\xi_n(t)]$ represents noise sources acting on a system. We will assume $\langle\xi_i(t)\rangle=0$.

We will concentrate on the situations where noise enters linearly into the Langevin equations (this simplifies the derivations significantly):

$$\boxed{\dot{x}_i(t)=f_i(x,t)+\sum_{j=1}^{m} g_{i,j}(x,t)\,\xi_j(t)}$$

Additionally, if for a given j all $g_{i,j}=\text{const}$, we speak about additive noise as the intensity of the action ξ_j on all $x_i(t)$ is independent of the actual state $x_i(t)$.
The opposite case, when $g_{i,j}$ is a function of state variable is called multiplicative noise.

6.2. Fokker-Planck versus Langevin equations.

Kinetic coefficients can be derived from a corresponding Langevin equation. For a given state variable $x(t)$ (for which the Langevin eq. is formulated) we need to determine the increment dx for the time interval dt (linear approximation) from the Langevin equation. Higher contributions vanish. Such a procedure gives the required connection between the Langevin and FPE.

The procedure shows that the increment dx can be represented in the sense of an Ito integral. It means that functions of the stochastic process x defining the increment dx at time moments t are determined. Therefore, and this is the purpose, these functions are independent of and not correlated with the actual increment dW_{dt} of the noise source at time t.

For all x at later times $s>t$ this statement is not true $\Rightarrow$ correlations between $x(s)$ and the increment dW_{dt} have to be taken into account. This is due to the fact that the behavior of the increments dW_{dt} (which constitute/form a typical trajectory $W(t)$) is discontinuous and the function $W(t)$ is at any time nonsmooth. Therefore, the limiting transition from the sum to the integral is not unique for $dt \to 0$ and depends on the choice of s within the interval $[t; t+dt]$ for which $x(s)$ is considered. As we have already noted: if $s=t$ there are no contributions from the correlations. If $s=t+dt$ we have to take the correlations into account $\Rightarrow$ there appears an additional term for the increment dx.

Therefore, every other choice of s different from $s=t$ needs further explanations. Let us now consider how to use the

Ito-formalism for an arbitrary s.

We start with a stochastic diff. equation:

$$\dot{x} = f(x,t) + \sqrt{2D}\, g(x,t)\, \xi(t), \quad (40)$$

where $x(t)$ is one-dimensional variable, $\xi(t)$ is Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t+\tau) \rangle = \delta(\tau)$.
The increment for a small time interval dt is:

$$dx_t = x(t+dt) - x(t) = f(x(s), s)\,dt + g(x(s), s)\sqrt{2D}\, dW_{dt}, \quad (41)$$

where dW_{dt} is the increment of the corresponding Wiener process at time t for the interval dt with $\langle dW_{dt}^2 \rangle = dt$.

The increment dx_t depends on a particular segmentation of the interval $[t, t+dt]$. This is due to the non-smoothness of white noise $\xi(t)$ and the resulting nondifferentiability of the Wiener process with increments dW_t. Let q be a constant $q \in [0,1]$ and $s = t + q\,dt = q(t+dt) + (1-q)t$.
We substitute s into (41) (+ expansion of f and g):

$$dx_t = \underbrace{f_t\,dt + q\frac{\partial f}{\partial x_t}dx_t\,dt}\; + \underbrace{g_t\sqrt{2D}\,dW_{dt} + q\frac{\partial g}{\partial x_t}dx_t\sqrt{2D}\,dW_{dt}}, \quad (42)$$

where all the functions are taken at time t.
Nevertheless, the increment dx_t still depends on the random value dW_{dt}. We re-insert dx_t into the right part of (42) and neglect third-order terms:

$$dx_t = \overline{f_t\,dt} + \overline{g_t\sqrt{2D}\,dW_{dt}} + q\frac{\partial g}{\partial x_t} g_t\, 2D\, dW_{dt}^2 +$$

$$+ \left(q\frac{\partial f}{\partial x_t} g_t\, dt + f_t g_t + q\frac{\partial g}{\partial x_t} f_t \right)\sqrt{2D}\, dW_{dt}\, dt \quad (43)$$

Stratonovich has shown that in addition to the two first terms the term $2D\,dW_{dt}^2$ also has an impact $\rightarrow$ it is a higher-order term which allows to determine dx_t more precisely.

In the short time limit $dt \to 0$ it can be replaced by its expectation $2D\,dt$ since the mean and the second moment of the deviations of the $2D\,dW_{dt}^2$ from $2D\,dt$ vanish. Both moments taken from the last term in (43) in the limit $dt \to 0$ are zero as well:

$$dx_t = f(x,t)\,dt + 2D\,dt\, q\, \frac{\partial g(x,t)}{\partial x}\, g(x,t) + \sqrt{2D}\, g(x,t) \cdot dW_{dt} \quad (44)$$

It follows from (44) that:

1) the increment is independent of the process $x(t')$ at previous times $t' < t$ => a white-noise-driven process is Markovian.

2) the increment is linear with respect to dW_{dt} and therefore erratic in its time evolution as dW_{dt}.