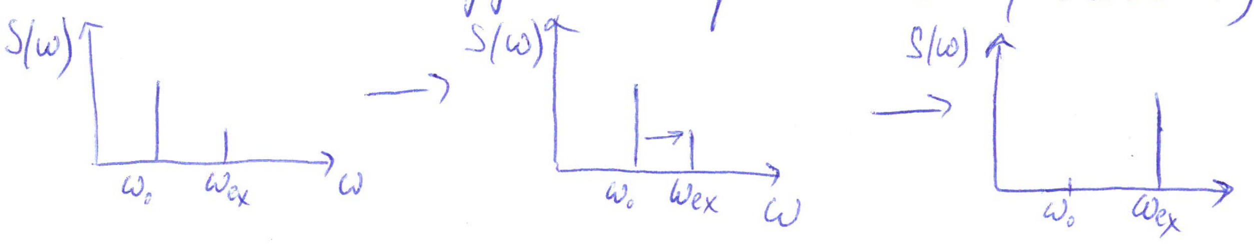
Synchronization

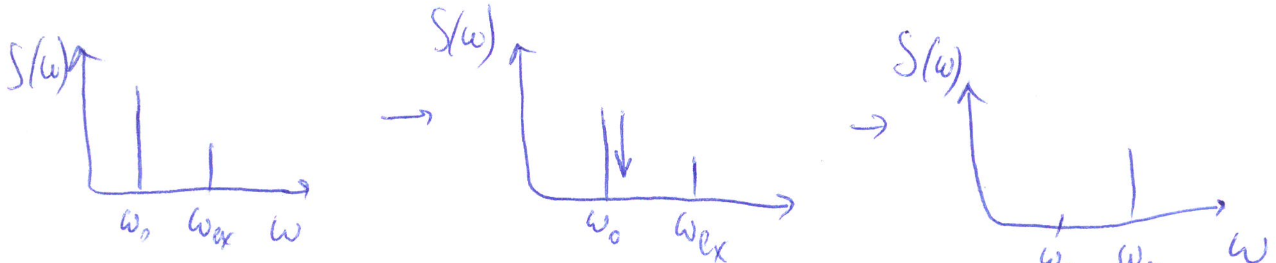
- one of the fundamental nonlinear phenomena observed in nature
- one of the basic mechanisms of self-organization of complex systems
- adjustment of some relations between characteristic times, frequencies or phases of two or more dynamical systems during their interaction

The classical theory of synchronization distinguishes between forced synchronization by an external periodic driving force and mutual synchronization of coupled oscillators.

Mechanisms: • phase / frequency locking

• suppression of intrinsic (inherent) frequencies





$\varphi(t) \rightarrow$ phase of the oscillator

$\Psi(t) \rightarrow$ phase of the external periodic force / another periodic oscillator

The condition of synchronization

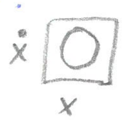
$$|m\varphi(t) - n\psi(t)| = const$$

m and n are integers. This condition defines the locking of two phases $\varphi(t)$ and $\psi(t)$ and requires that the phase difference should be constant.
Synchronization can be also defined as frequency locking, provided that the frequency of the oscillator and the frequency of the driving force are in rational relation.
The simplest case is 1:1 synch: $m=1$; $n=1$.

What is the impact of noise? The theory shows that noise counteracts synchronization: under the influence of noise synch. occurs only for a limited time interval. On the other hand noise can induce new regimes, for example, noise-induced switchings.
- signals are not periodic
- power spectrum is not discrete (may not contain peaks at any distinct frequencies.

Nevertheless, the concept of phase and freq. locking can be applied to this case as well.

Phase → this term was originally introduced for harmonic processes $x(t) = A\cos(\omega_0 t + \varphi_0)$.
If $x(t)$ is strictly periodic with a constant amplitude A ⟹ $\boxed{\varphi(t) = \omega_0 t + \varphi_0}$ is istantaneous phase. The amplitude and phase can be found from $x(t)$ and a second conjugated variable → its time derivative $\dot{x}(t)$. This definition corresponds to a transition to polar coordinates:



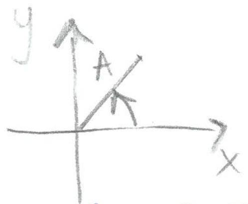
radius A and the angle φ:

$$A = \sqrt{x^2 + \frac{\dot{x}^2}{\omega_0^2}}, \quad \tan\varphi = -\frac{\dot{x}}{x\,\omega_0},$$

where ω_0 is the frequency.

A similar approach is the analytic continuation of a periodic signal into the complex plane. Instead of real signal $x(t)$ we introduce:

$$z(t) = x(t) + i y(t) = A \exp i(\omega_0 t + \varphi_0) = A[\cos(\omega_0 t + \varphi_0) + i \sin(\omega_0 t + \varphi_0)],$$ which is now complex.



The harmonic oscillation is pictured as a rotation of the vector A with constant angular velocity ω_0 in the complex plane. The phase of oscillations corresponds to the angle of the vector A; and the angle φ_0 at $t=0$ is the initial phase.

However, these definitions can not be applied directly and used in nonlinear dissipative systems, since the oscillations in such systems are not harmonic. How to proceed in more complex situations? How to define the phase for a noisy dynamical system? How to introduce the phase for a chaotic system?

Let us consider the dynamics of the periodically driven van der Pol oscillator:

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + \omega_0^2 x = a\cos(\Omega t)$$

We consider the case of small nonlinearities $0 < \varepsilon < 1$ and introduce instantaneous phase and amplitude using amplitude and phase description (see Lecture 14) with $\Phi(t) = \Omega t + \varphi(t)$. Conditions for slow amplitude and phase evolution are $\omega_0 \approx \Omega$, small ε (sinusoidal oscillations) and that the amplitude of the periodic force also scales with ε.

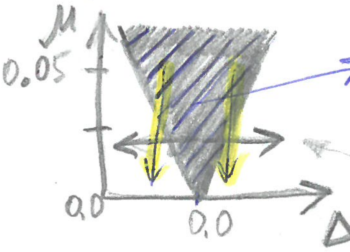
↓ gives equations for A and φ

$$\begin{cases} \dot{A} = \dfrac{\varepsilon A}{2}\left(1 - \dfrac{A^2}{A_0^2}\right) - \mu \sin\varphi, \\ \dot{\varphi} = \Delta - \dfrac{\mu}{A}\cos\varphi. \end{cases}$$

$\mu = \dfrac{a}{2\Omega}$ — effective amplitude of ext. f.; Δ is frequency mismatch

$\Delta = (\omega_0^2 - \Omega^2)/2\Omega \approx \omega_0 - \Omega$

$A_0 = 2$ is the amplitude of the stable cycle in the autonomous case.



(*) synchronization region

The synchronous regime is destroyed by increasing the frequency mismatch (absolute) or decreasing the effective amplitude of the external force.

Synchronization as phase and frequency locking

Synchronization can be well represented by the long-term behaviour of the instantaneous frequency of the oscillator = time derivative of the instan. phase:

$$\omega(t) = \frac{d}{dt}\Phi(t)$$

This additional reduction of the dynamics, allowing a consideration of the phase dynamics only, requires further assumptions on the amplitude dynamics. It will be valid for the interesting situation of a small amplitude of the external periodic force: $\mu/(A_0\varepsilon) \ll 1$ or $\varepsilon \gg a/(2\Omega A_0)$. This condition holds in synch. region (*)

One can consider the equation for the phase separately:

$$\boxed{\frac{d}{dt}\varphi = \Delta - \frac{\mu}{A_0}\cos\varphi}, \qquad \mu = a/2\Omega$$

$$\Delta \approx \omega_0 - \Omega$$

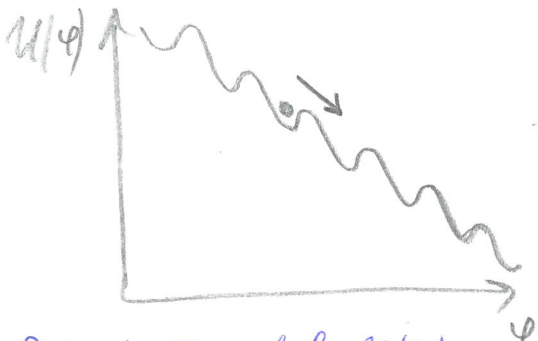
This equation is one of the canonical ones in the theory of phase synchronization. It can be rewritten in a potential form:

$$\dot{\varphi} = -\frac{dU(\varphi)}{d\varphi}$$

with the potential $U(\varphi) = -\Delta\cdot\varphi + \frac{\mu}{A_0}\sin\varphi$. Therefore, the dynamics of the phase difference φ can be seen as the motion of an overdamped particle in the tilted potential $U(\varphi)$. The detuning parameter Δ determines the slope of the potential and μ/A_0 gives the height of the potential barriers. For $\Delta < \mu/A_0$ the minima of the potential

$$\varphi_k = \arccos(\Delta\cdot A_0/\mu) + 2\pi k,$$

$k = 0, \pm1, \pm2, \ldots$ correspond to synchronization as the instantaneous phase difference remains constant in time.

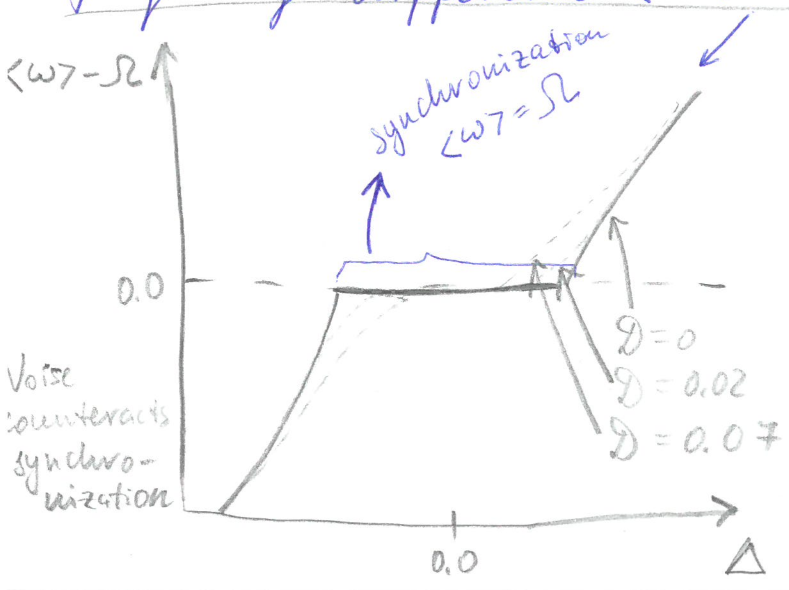


Potential profile $U(\varphi)$ in the case of phase locking for $\Delta \neq 0$.

The instantaneous frequency is constant and matches the driving force frequency in the regime of synchronization. Otherwise it changes in time and we have to calculate the mean frequency as

$$\langle\omega\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \omega(t)\,dt.$$

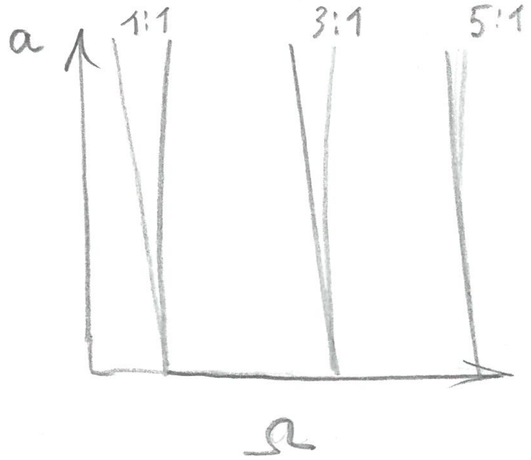
The dependence of the frequency difference $\langle\omega\rangle - \Omega$ on detuning Δ for $D=0$ and $D\neq 0$ shows that in the regime of synchronization the mean frequency coincides with the external frequency Ω in a finite range of Δ. The plateau corresponds to the synchronization region. Outside this region $\langle\omega\rangle$ differs from the



external frequency and two-frequency oscillations occur. If we increase the detuning further, then high-order regimes of synchronization can occur. To study these regimes we introduce the ratio of the driving frequency to the mean frequency of the oscillator

$$\theta = \frac{\Omega}{\langle \omega \rangle}.$$

This ratio shows how many periods of external force are within one period of the oscillator. Up to now we have only considered the regime of 1:1 synchronization when $\theta = 1$.



These regions are called "Arnold tongues".