Lecture 4

09.11.2016

Bifurcations

deterministic

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$$

(parameters)

stochastic

(noise intensity is the parameter)

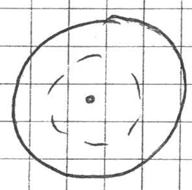
$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu}) + \xi(t)$$

$\xi(t)$ – noise

What do we have in the determ. case?

Attractors

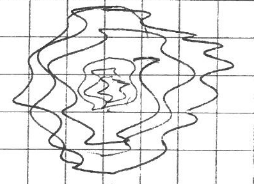
bistability: two attractors



What do we have in the stochastic case?

one invariant set of trajectories

dependence on initial conditions disappears



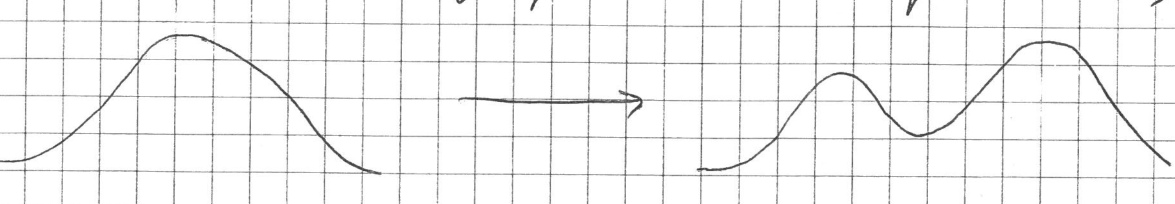
one can not distinguish between attractors

How to characterize the dynamics of stochastic systems?
How to track the dynamical changes in the stoch. case?

Stochastic bifurcations

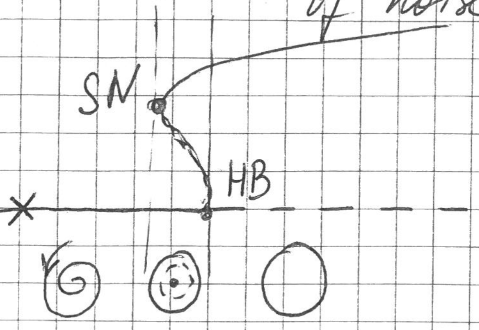
- Phenomenological (P-bifurcation) stochastic bifurcation

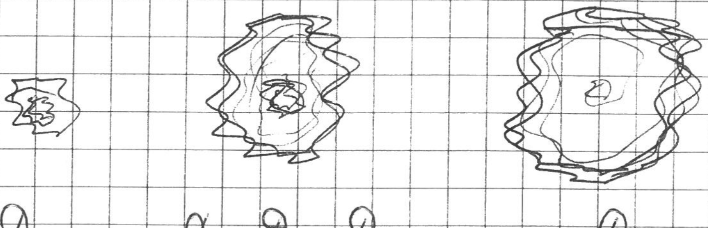
a qualitative change of the stationary probability distribution; (transition from a unimodal distribution to a bimodal → change of the number of maxima).



L. Arnold, Random Dynamical System. Springer, Berlin (2003)

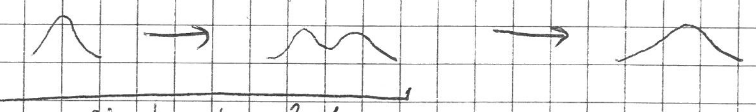
Example subcritical Hopf bifurcation in the presence of noise





$\mathcal{D}_1$ $\quad$ $\mathcal{D}_1 < \mathcal{D}_2 < \mathcal{D}_3$ $\quad$ $\mathcal{D}_3$

Stochastic P-bifurcation



Stochastic Bif.
P-bifurcation

- Dynamical bifurcation (stochastic D-bifurcation) a change of stability of trajectories belonging to a certain set with a given invariant measure (a sign change of one of the Lyapunov exponents).

Stochastic bifurcation → P-bifurcation; → D-bifurcation; → P-bifurcation + D-bifurcation

---

Random process → discrete; → continuous

$x(t)$ – random process

$p(x)$ – probability density (probability distribution)

$p(x)$ assigns a probability to each measurable subset of the possible outcomes of a random experiment.

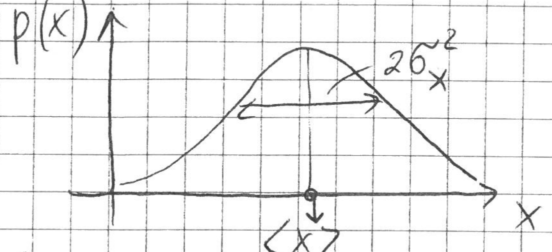
$p(x)$ → multivariate (joint)
gives the probability of a random vector → a set of two or more random variables.

$p(x)$ → univariate
gives probabilities of a single random variable

- $p(x) \geq 0$
- $\int_{-\infty}^{\infty} p(x)\,dx = 1$
- $P(\alpha < X < \beta) = \int_{\alpha}^{\beta} p(x)\,dx$

Examples: Gaussian distribution, Poisson distribution

<u>mean value</u> $\langle x(t) \rangle = \int_{-\infty}^{\infty} x\,p(x)\,dx$



σ_x^2 defines the width of the distribution

<u>fluctuation</u> $\Delta X = X - \langle x \rangle$ deviation from the mean

<u>variance</u> $\sigma_x^2 = \int_{-\infty}^{\infty} (x - \langle x(t) \rangle)^2 p(x)\,dx = \langle x^2 \rangle - \langle x \rangle^2$

<u>correlation function</u>

$$\langle \Delta X(t_1)\, \Delta X(t_2) \rangle = \iint_{-\infty}^{\infty} (x(t_1) - \langle x(t_1) \rangle)(x(t_2) - \langle x(t_2) \rangle) \cdot p(x_1, t_1, x_2, t_2)\,dx_1\,dx_2 =$$

$$= \langle x_1 x_2 \rangle - \langle x_1 \rangle \langle x_2 \rangle$$

$x_1 = x(t_1)$ | $\Psi_x(t_1, t_2)$

$x_2 = x(t_2)$

<u>Gaussian noise</u> is a stochastic process with a probability density function equal to that of the normal distribution.

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

$\rightarrow$ probability density function of a random variable $n(t)$.

A special case of Gaussian noise is <u>white</u> Gaussian noise, in which the values at <u>any</u> pair of times are statistically independent (uncorrelated).

$$\langle n(t)\, n(t+\tau) \rangle = \delta(\tau), \quad \langle n(t) \rangle = 0$$

<u>Uniform</u> power for all the frequencies.

Noise $\rightarrow$ white, colored

If noise (intensity) does not depend on X and is a deterministic function of time or constant, the noise is additive. If the noise intensity depends on X $\rightarrow$ the noise is multiplicative.