7. Coherence resonance

Lecture 11

11.01.2017

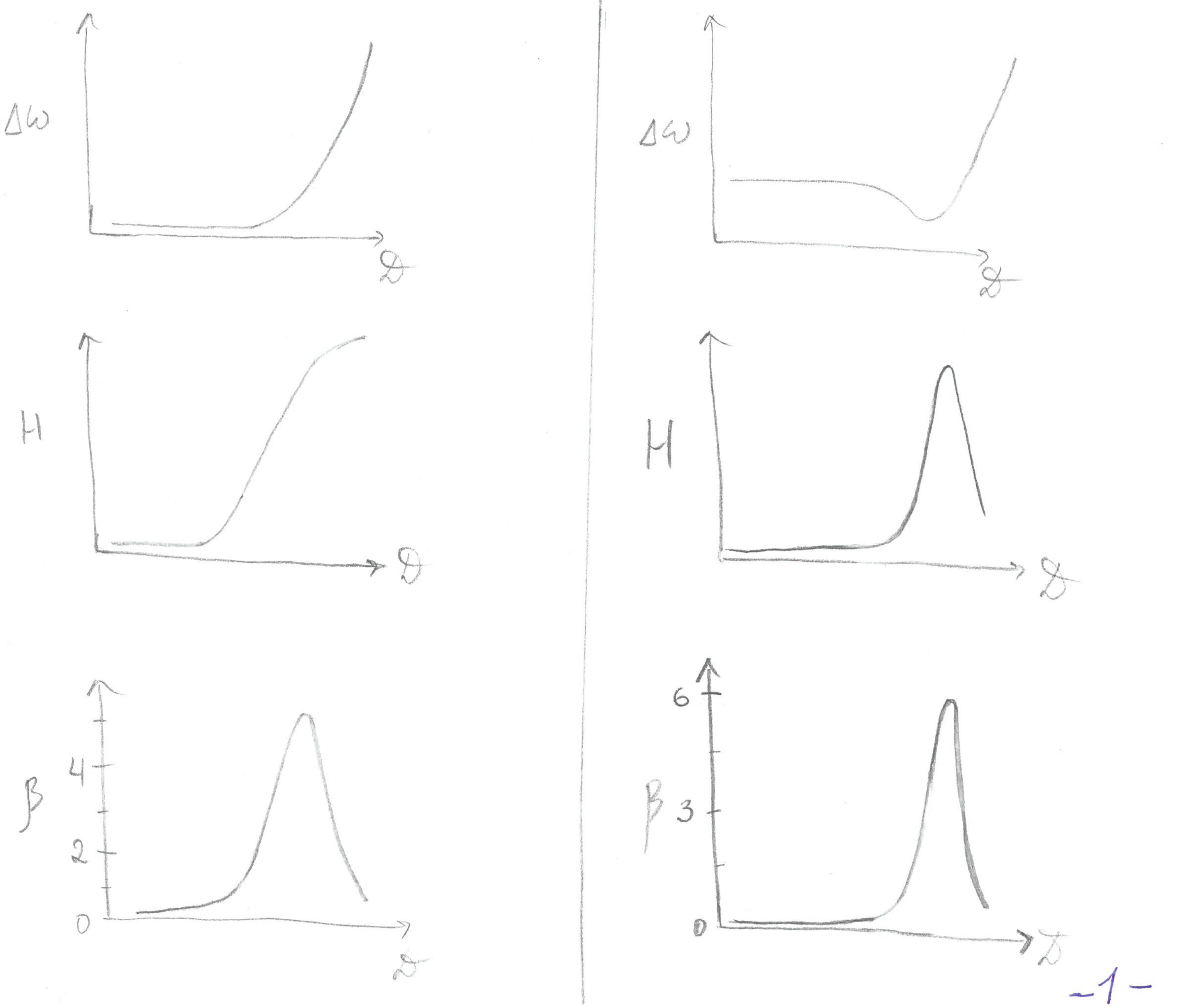
7.3 Coherence resonance in non-excitable systems

[Ushakov et al. PRL 95, 2005]

7.3.1 Stuart-Landau oscillator (+ semiconductor laser with delayed optical feedback)

signal-to-noise ratio (SNR) $\beta = \frac{H}{\Delta\omega/\omega_p}$

Supercritical Hopf bif. | Subcritical Hopf bif.



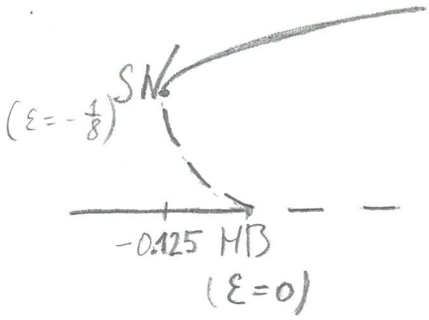
Both bifurcations demonstrate resonance-like behaviour: width and peak hight exhibit the same qualitative behaviour. For the <u>supercritical Hopf bifurcation</u> $\Delta\omega$ stays practically constant for small noise and increases as $\sqrt{D}$ at larger D. Instead, the peak hight grows initially like $H \sim D$ and saturates for stronger noise. In the <u>subcritical case</u> for small noise levels $\Delta\omega$ is constant as in the supercritical case. However, the spectral width is a nonmonotonic function with a distinct minimum at a certain noise level. Moreover, the peak hight has a clear maximum. Therefore, in the supercritical case the increase of the signal-to-noise ratio is produced by the spectral peak hight, that is, by an increase of the oscillation amplitude. <u>The width is initially only slightly affected</u>, but increases steeply for stronger noise, weakening the coherence. The resonance behaviour originates from the competition between the growth of hight and width. In contrast, for the subcritical case, the spectral width itself exhibits a minimum → the noise improves, indeed, the quality factor as well as the temporal coherence of the oscillations. Therefore, in a strict sense, only subcritical HB provides the truly CR.

7.3.1 A model of a generalized Duffing-van der Pol oscillator [Zakharova et al. PRE 2010]

$$\boxed{\ddot{x} - (\varepsilon + x^2 - x^4)\dot{x} + x + \beta x^3 = \sqrt{2D}\, n(t)}, \quad \beta \geq 0$$

$n(t)$ is Gaussian white noise: $\langle n(t)\, n(t+\tau)\rangle = \delta(\tau)$, $\langle n(t)\rangle = 0$; D is the noise intensity; β defines anisochronicity of oscillations: for $\beta = 0$ the system is isochronous → the frequency of the oscillations does not depend on the amplitude. For $D=0$ the system is characterized by bistability:

$-\frac{1}{8} < \varepsilon < 0$. There are two attractors in the phase space: a stable focus and a stable limit cycle.



SN - saddle-node bifurcation of limit cycles occurs for $\varepsilon=-\frac{1}{8}$.

HB - Hopf bifurcation (subcritical) is observed for $\varepsilon=0$. ε is a bifurcation parameter.

Analytical approach: we assume that noise intensity is small and that the fluctuations of the amplitude and phase are "slow" stochastic processes $\Rightarrow$ they remain unchanged during the period $T_0 = 2\pi$ $(\omega_0 = 1)$.

We make change of variables:

$$x(t) = a(t)\cos[t+\varphi(t)], \quad \dot{x}(t) = -a(t)\sin[t+\varphi(t)]$$

$a(t)$ - instantaneous amplitude, $\varphi(t)$ - instantaneous phase

We substitute the new variables into the equation describing Duffing-van der Pol oscillator and average the equations over the period of oscillations [see details of the method in R.L. Stratonovich, Selected Topics in the Theory of Random Noise 1963, Vol. 1 and 2]. We obtain stochastic equations for the slow random variables:

$$\boxed{\begin{aligned} \dot{a} &= \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right)a + \frac{D}{2a} + \sqrt{D}\,n_1(t), \\ \dot{\varphi} &= \frac{3\beta}{8}a^2 + \frac{\sqrt{D}}{a}\,n_2(t), \end{aligned}}$$

where $n_1(t)$ and $n_2(t)$ are independent sources of Gaussian white noise. From these equations one can derive the amplitude of the stable limit cycle for $\mathcal{D}=0$: $a_0 = \sqrt{1+\sqrt{1+8\varepsilon}}$.

An important observation is that $\dot{a}$ does not depend on $\varphi \Rightarrow$ we can consider $p(a,t)$ rather than $p(a,\varphi,t)$ joint prob. density for φ and a. Therefore, we further analyze the equation for the instan. amplitude separately:

$$\boxed{\dot{a} = \underbrace{\left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right)a + \frac{\mathcal{D}}{2a}}_{f(x,t)} + \sqrt{\mathcal{D}}\, n_1(t)}$$

Langevin equation

$$\boxed{\dot{x} = f(x,t) + \sqrt{2B}\, n_s(t)}$$

The diffusion process described by this equation is characterized by the drift coefficient

$$A(a) = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right)a + \frac{\mathcal{D}}{2a}$$

and by the diffusion coefficient $B = \frac{\mathcal{D}}{2}$.

[see kinetic coefficients (K_1 and $K_2 \rightarrow$ eq. (46) and (47)), Langevin equation (40) and FPE (48) in Lecture 9]

For $\mathcal{D} \neq 0$ the probability density $p(a,t)$ of the instantaneous amplitude satisfies the Fokker-Planck equation for one-dimensional process $a(t)$:

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[A(a)\, p(a,t)\right] + B\,\frac{\partial^2 p(a,t)}{\partial a^2}$$

We substitute $A(a)$ and B into the FPE:

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[\left(\frac{\varepsilon a}{2} + \frac{a^3}{8} - \frac{a^5}{16} + \frac{\mathcal{D}}{2a}\right)p(a,t)\right] + \frac{\mathcal{D}}{2}\,\frac{\partial^2 p(a,t)}{\partial a^2}$$

The stationary solution of FPE for one-dimensional process $a(t)$:

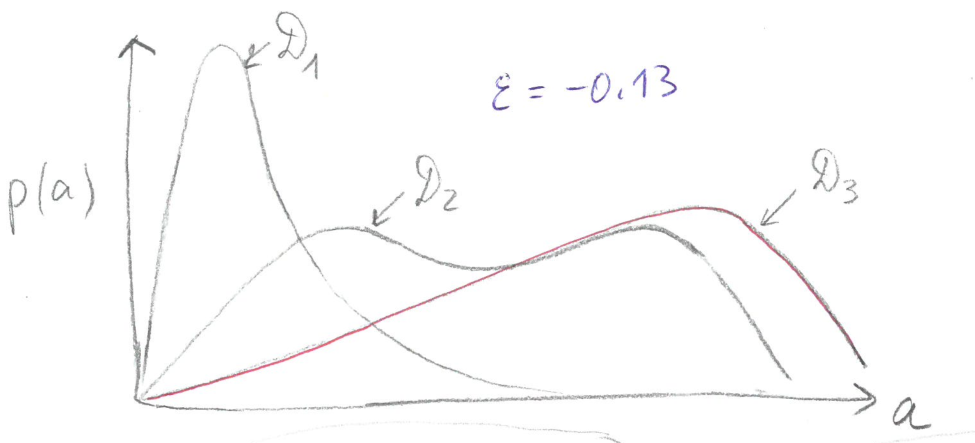
$$p_1^{st}(a) = \frac{C_0}{B(a)} \exp\left[\int \frac{A(a)}{B(a)}\, da\right],$$

where $C_0 = \text{const}$. This formula can be found in any book on stoch. processes. Here we get:

$$p(a) = N \cdot a \cdot \exp\left[-\frac{1}{48D} a^2 (a^4 - 3a^2 - 24\varepsilon)\right],$$

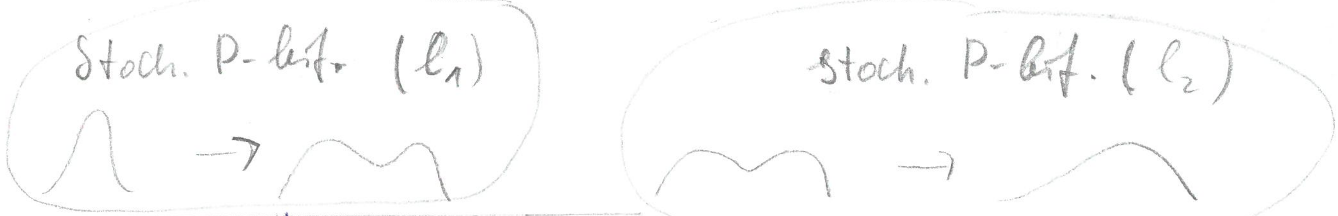
where N is normalization constant.

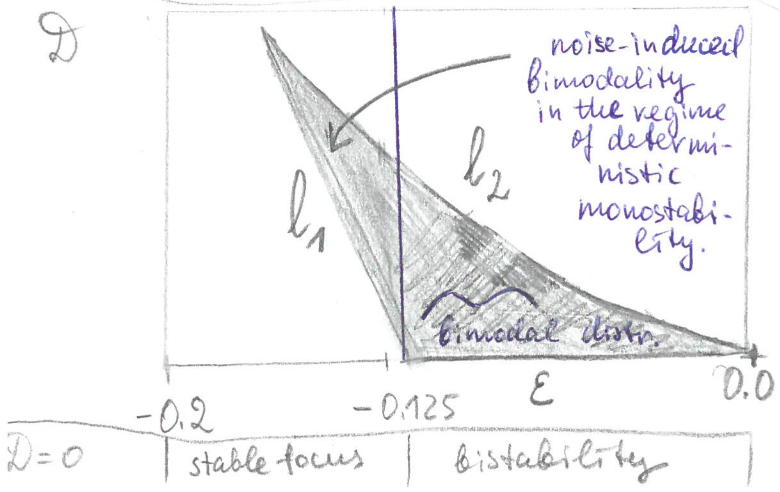
Since a does not depend on φ the shape of the distribution is the same for isochronous and and anisochronous oscillators.



$D_1 < D_2 < D_3$

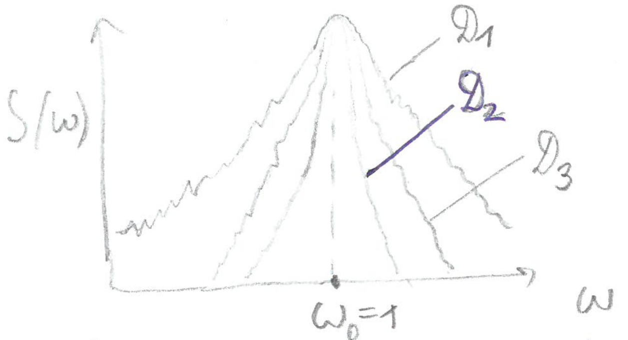
Stochastic P-Bifurcations!





In the shaded region the stationary amplitude distribution is bimodal. For $D=0$ the region of bistability is in the regime between HB ($\varepsilon=0$) and SN ($\varepsilon=-0.125$). In the presence of noise the bimodality region shifts to the smaller values of ε and becomes narrower.

<u>Power spectral density for $\beta = 0$</u>



The spectral line width becomes minimal for an intermediate noise intensity.
The effect of CR is connected with stochastic bifurcations: coherence resonance is the most pronounced for noise amplitudes within a region of a bimodal amplitude distribution $\Rightarrow$ CR occurs after the stochastic P-bifurcation on line l_1.