Last time we have shown that a white-noise-driven process $x(t')$ is Markovian and its increment is erratic in its time evolution. Moreover, the increment is linear with respect to dW_{dt}.

What is the number of intersections?

For example, one may ask about the number of intersections of $x(t)$ with a given boundary value x_0 (if IC is in the vicinity of x_0). In the limit $dt \to 0$ the linear dependence on dW_{dt} guarantees unbounded growth of the number of intersections.

The increment in (44) is an Ito-formalization of the increment dx.

Summing up the increments from 0 to t we get the stochastic Ito-integral with a certain segmentation:

as defined in (*) → Lecture 8

$$x(t) = x(0) + \int_0^t ds \left[f(x,s) + 2\vartheta q \frac{\partial g(x,s)}{\partial x} g(x,s) \right] + \sqrt{2\vartheta} \int_0^t dW_s \, g(x,s) \quad (45)$$

Since W_t is continuous we obtain continuity also for $x(t)$.

The kinetic coefficients of the FPE that are conditional moments of $x(t)$ per unit time can be obtained from (45):

$$K_1(x,t) = \frac{d\langle x\rangle}{dt} = f(x,t) + 2\mathcal{D}q \frac{\partial g(x,t)}{\partial x} g(x,t) \quad (46)$$

and

$$K_2(x,t) = \frac{1}{2}\frac{d\langle x^2\rangle}{dt} = \mathcal{D} g^2(x,t) \quad (47)$$

The high-order coefficient vanishes due to the properties of the Wiener process (32). Therefore, the processes defined by the Langevin equation with Gaussian white noise are Markovian diffusion processes.

Thus, FPE for the transition probability density $p_2 = p_2(x,t\,|\,x_0,t_0)$ corresponding to the Langevin eq. (40) is

$$\boxed{\frac{\partial}{\partial t} p_2 = -\frac{\partial}{\partial x}\left(f(x,t) + 2\mathcal{D}q\frac{\partial g(x,t)}{\partial x} g(x,t)\right)p_2 + \mathcal{D}\frac{\partial^2}{\partial x^2} g^2(x,t)\, p_2} \quad (48)$$

In the same way we can obtain kinetic coefficients in the multi-dimensional case:

$$\dot{x}_i = f_i(x_1,\dots,x_n) + \sum_{j=1}^{m} g_{ij}(x_1,\dots,x_n)\,\xi_j(t), \quad (49)$$

where we have m white noise sources with zero mean and $\langle \xi_i(t)\,\xi_j(t+\tau)\rangle = 2\mathcal{D}_{ij}\,\delta(\tau)$ (50)

From expressions for the conditioned moments of the increments dx_i one obtaines:

$$K_1^i = f_i + 2q \sum_{j,k,l} D_{j,l} \frac{\partial g_{i,j}}{\partial x_k} g_{k,l} , \quad (51)$$

$$K_2^{i,j} = \sum_{k,l} D_{k,l}\, g_{i,k}\, g_{j,l} \quad (52)$$

Again the substitution of these coefficients into (39) gives the forward time evolution operator which fully defines the stoch. dynamics of the higher dimensional process.

general form of forward FPE in the case of many dyn. variables

This type of the evolution operator with the drift and diffusion term was first formulated for the motion of a Brownian particle in the phase space by Fokker and later generalized by Planck. The first math. formulations go back to Kolmogorov. All these names can be found in the literature in relation with equation (39) → F-P-equation.

It is important to note that the last term in (45) is always defined as Ito integral ((*) in lecture 8). Therefore, the relation between the kinetic coefficients and the Langevin equation in the case of multiplicative noise depends on the choice of the partition, i.e. on the constant q (see lecture 8). There is no universal approach to select this parameter.

g(x,s) is a function (not constant)

$q \in [0,1]$ and $s = t + q\,dt$

It depends on how the limit to white noise is taken compared to $dt \to 0$.

One can show that $q = \frac{1}{2}$ corresponds to the case when white noise is assumed to be the small time limit of colored noise with correlations going to zero.

This choice of q corresponds to Stratonovich calculus for treating stochastic Ito-integrals. The advantage of it is the fact that the relation between the conditional moments and the Langevin equation is invariant with respect to a variable transformation. For all other values of q the transformation rules have to be modified, even for the case $q=0$. In mathematical literature the preference is given to $q=0$, which is called the Ito calculus. In this case the increment dx_t relies on the random impact of dW_t at t only (since $s = t + q\,dt$, $q=0$). It can be justified if the continuous process $x(t)$ is the model of some discrete-valued process and the white noise limit is taken as continuous approximation.

Ito calculus: $q=0$ ← math.

Stratonovich calculus: $q = \frac{1}{2}$ ← physics