Synchronization in the presence of noise. Effective synchronization.

Lecture 16

15.02.2017

Last time we have mainly focused on the deterministic case $\rightarrow$ no random perturbations are present. However, noise is unavoidable in any real system in the form of

- natural (or internal) fluctuations
- random perturbations of the environment

The introduction of the phase in a noisy oscillating system requires a probabilistic approach.

one can use the amplitude and phase description approach $\Rightarrow$ the istantaneous amplitude and phase change into stochastic variables since $x(t)$ and $\dot{x}(t)$ are stochastic

with noise taken into account the amplitude and the phase dynamics is described by stochastic differential equations including a noise term $\xi(t)$.

To extract the information from stochastic dynamics we have to calculate the moments of $A(t)$, $\varphi(t)$ and $\omega(t) = \dot{\varphi}(t)$ or consider the transition probability density $p(A, \varphi, t \,|\, A^*, \varphi^*, t_0)$ which is sufficient for Markovian approximations.

$p(A, \varphi, t \,/\, A^*, \varphi^*, t_0)$ – conditional probability to observe the amplitude A and the phase φ at time t if started at time t_0 with A^* and φ^*.

In noisy systems the phase $\varphi(t)$ as well as the difference with respect to the external driving $\psi(t) = \varphi(t) - \Omega t$ performs motion similar to Brownian particle in the potential $\mathcal{U}(\psi)$. The stochastic process $\psi(t)$ can be decomposed into two parts

$\varphi(t)$

a deterministic part given by its mean value (or the mean value of the instantaneous frequency)

and a fluctuating part characterized, for example, by the diffusion coefficient around its mean value.

Synchronization as a fixed relation between two phases is always interrupted by randomly occurring abrupt changes in the phase difference, also known as phase slips

$\Downarrow$

In noisy oscillating systems the notion of synchronization must be mathematically expressed by relations and conditions between the moments of the fluctuating phase or its corresponding probability density.

Langevin Equation Description

The stochastic force $\xi(t)$ is added to the deterministic diff. eq. of a periodically driven van der Pol oscillator:

$$\ddot{x} - \varepsilon(1-x^2)\dot{x} + \omega_0^2 x = a\cos\Omega t + \sqrt{2D_0}\,\xi(t)$$

One can obtain reduced equations for the stochastic amplitude and phase difference:

$$\dot{A} = \frac{\varepsilon A}{2}\left(1 - \frac{A^2}{A_0^2}\right) - \mu\sin\varphi + \frac{D}{A} + \sqrt{2D}\,\xi_1(t)$$

$$\dot{\varphi} = \Delta - \frac{\mu}{A}\cos\varphi + \frac{\sqrt{2D}}{A}\xi_2(t).$$

ξ_1 and ξ_2 are statistically independent Gaussian noise sources: $\langle \xi_i(t)\,\xi_j(t+\tau)\rangle = \delta_{ij}\delta(\tau)$ and $\langle \xi_i(t)\rangle = 0$ for $i,j = 1,2$. $D = D_0/(2\Omega^2)$ is the common intensity of the two transformed noise sources $\xi_{1,2}$.

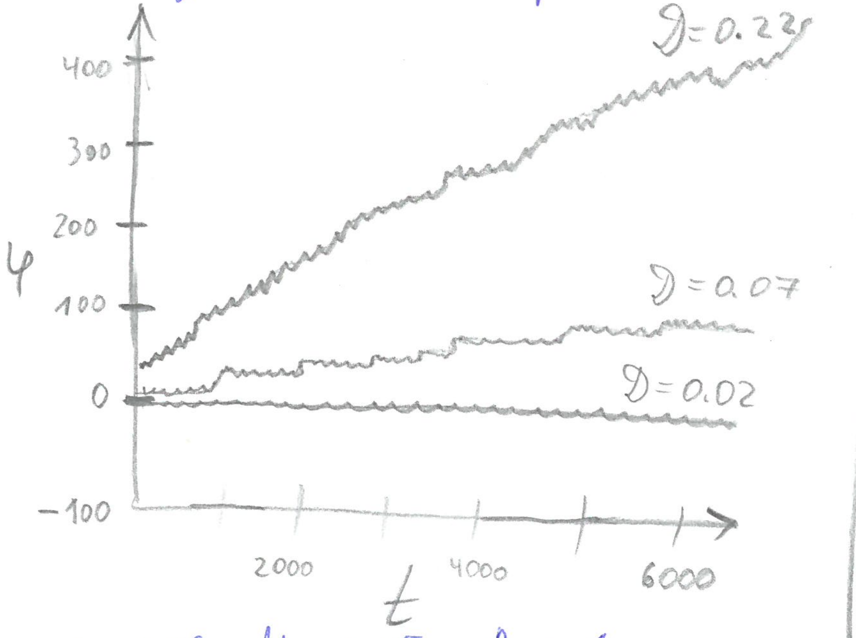
As we did last time (synchronization for $D=0$) let us consider again the most interesting situation corresponding to small amplitudes of the external force. For small noise $D \ll \varepsilon A_0^2/2$ and weak external force/signal the probability distribution is narrowly centered at $A \approx A_0 \Rightarrow$ the amplitude is close to its unperturbed value A_0. This gives us the opportunity to consider the phase equation separately by substitution of A_0 instead of A:

$$\dot{\varphi} = \Delta - \frac{\mu}{A_0}\cos\varphi + \frac{\sqrt{2D}}{A_0}\xi_2(t)$$

The dynamics of the phase difference φ can be viewed as the motion of an overdamped Brownian particle in the tilted potential $U(\varphi)$ with the slope defined by $\Delta \rightarrow$ detuning. The parameter $\mu/A_0 = \Delta_s$ gives the height of the potential barriers.

The presence of noise leads to the diffusion of the instantaneous phase difference in the potential $U(\varphi) \rightarrow \varphi(t)$ fluctuates for a long time inside a potential well (which means phase locking) and rarely makes jumps from one potential well to another (displays phase slips) changing its value by 2π.

One can integrate the equation for $\varphi(t)$ numerically and get time series of phase difference for different values of noise intensity D. For small noise intensities ($D=0.02$) the instantaneous phase difference remains bounded during a long observation time. The increase of the noise intensity leads to a decrease of the average duration of residence times inside a potential well and causes the hopping dynamics of the phase difference ($D=0.07$).



$\Delta = 0.06,\ \mu = 0.15,\ A_0 = 1$

Although the phase locking times ($\varphi = \mathrm{const}$) are clearly seen, the mean value of the phase difference increases in time. For a large slope (detuning) and for a small value of the periodic force amplitude, the jumps from one metastable state to another become very frequent and the duration of phase locking segments becomes very short. This leads to the grows of phase difference (see the curve $D=0.22$) causing a change of the mean frequency of oscillators.

Phase description

Now we deal with aperiodic process => new concepts are required. There are many different ways to introduce the phase => The choice of the definition depends on the dynamical system and the signals considered. Here two characteristics are important: mean angular velocity $\langle\omega\rangle$ and effective diffusion (constant) coefficient D_{eff}.

$$\langle\omega\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \frac{d\varphi(t)}{dt}\, dt = \lim_{T\to\infty} \frac{1}{T}\left(\varphi(t_0+T) - \varphi(t_0)\right)$$

$$D_{eff} = \frac{1}{2} \frac{d}{dt}\left[\langle\varphi^2(t)\rangle - \langle\varphi(t)\rangle^2\right]$$

phases in the analytic signal representation

One concept has been successfully applied to a description of chaos synchronization. For a signal $x(t)$ one constructs an analytic signal $w(t)$ in the complex plane by

$$w(t) = x(t) + i y(t) = A(t) e^{i\varphi(t)}$$

Then by definition the instantaneous amplitude and phase straightforwardly follow:

$$A(t) = \sqrt{x^2(t) + y^2(t)},$$

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$$\varphi(t) = \arctan\left(\frac{y}{x}\right) + \pi k, \quad k = 0; \pm 1; \pm 2, \dots$$

The instantaneous frequency is defined as

$$\omega(t) = \frac{d}{dt}\varphi(t) = \frac{1}{A^2(t)}\left[x(t)\dot{y}(t) - y(t)\dot{x}(t)\right]$$

The main question is how to define $\underline{y(t)}$.

Often $\boxed{\text{Hilbert transform}}$ is used:

$x(t) \to H[x(t)]$

$H[H[x(t)]] = -x(t)$

$$y(t) = H[x] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau =$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{x(t-\tau) - x(t+\tau)}{\tau}\, d\tau.$$

Properties As a linear transformation $H[x]$ obeys several useful properties

* every Hilbert transform of a linear superposition of two signals is the superposition of the separate Hilbert transforms

* if the time of the signal is shifted by some amount, it also shifts the argument of the Hilbert transform.

* the Hilbert transform of a Hilbert transform gives the negative original signal.

* even functions give odd Hilbert transforms and vice versa.

* The original signal and the Hilbert transform are orthogonal.

* The full energy of the original signal, the integral of $x^2(t)$ over all times, is equal to the energy of the transformed one.

There are also other way to introduce the phase.

Having introduced the phase one can develop a concept of phase-frequency synchronization of both chaotic oscillations and stochastic processes.

A generalized definition of phase synchronization which can be applied to these cases is based on the restriction:

$$\lim_{t \to \infty} | n \varphi_1(t) - m \varphi_2(t) | < const$$

The basic indication of synchronization is the appearance of the oscillatory regime with a constant and rational "winding number" (or rotation number): $\Theta = m:n$ which holds in some finite region of the system's parameter space. This region is called the synchronization region and is characterized by the effects of phase and frequency locking.

<u>Frequency locking</u>

means a rational ratio of two initially independent frequencies $\omega_1/\omega_2 = m:n$ everywhere in synch. region

<u>Phase locking</u>

means that the instantaneous phase difference is constant in the synchronization region ($\dot{\varphi} = 0$, $\varphi_{st} = const$).

The influence of noise on an oscillator leads to destruction of the synch. regime in the sense of the above-given definition. However, if the noise intensity is small, the main properties of synch. may survive.

Due to phase diffusion the definition of synchronization in the presence of noise appears to be "blurred" => the conditions of synch. should be defined in a statistical way by using the notion of <u>effective synchronization.</u> This can be done by imposing a certain restriction on some statistical measures of corresponding stochastic processes. In particular, a definition of effective synchronization can be based on the following items:

* the stationary prob. density of the phase difference
In this case the peak in $P_{st}(\varphi)$ should be well expressed in comparison with the uniform distribution.

* the mean frequency. This should match the driving frequency (up to some small statistical error).

* the effective diffusion (coefficient) constant. This measure should be <u>small enough</u> so that the phase locking segments are much longer than one period of the external force. In other words, this restriction requires that the <u>oscillator phase is locked during a considerable number of periods</u> of the external signal:

$$D_{eff} \leq \frac{\Omega}{n}, \quad n \gg 1$$

$n \rightarrow$ number of periods of the ext. force.