5. Basic concepts of stochastic dynamics.

Deterministic case: a phase traj. is uniquely defined by I.C.

Stochastic case: the state $x(t)$ is not uniquely mapped in time.

The math. formulation can be given by **stochastic differential equations (SDE)** explicitly including the random sourses:

$$\dot{x}_\xi = f(x_\xi, \xi(t)) \qquad (1)$$

$\xi(t)$ is a temporal sequence of random numbers generated due to some rule (the time evolution depends on the generated number).

$x_\xi(t) \to x_\xi(t+dt)$: the mapping depends on the concrete choice of $\xi(t)$ $\Rightarrow$

$\Rightarrow$ $x_\xi(t)$ is a **stochastic process**.

Due to the existence of the random forces $\xi(t)$ which perturb the system, different measurements of phase trajectories starting from the same I.C. will give different realizations of $x_\xi(t)$. Therefore, as in statistical physics, we have to consider an **ensemble** of N different phase trajectories (instead of a single one).

This ensemble is determined in its turn by an ensemble of realizations of random sources perturbing the system $\xi_N(t)$.

The statistical ensemble of $N \to \infty$ realizations defines a **stochastic process**.

The alternative definition of a stochastic process $x(t)$ can be made in terms of probability distributions (or probability densities).

We measure realizations of the process $x_1, x_2, \ldots, x_n$ at times $t_1, t_2, \ldots, t_n$ and give the joint probability density for their occurence:

$$p(x_1, t_1;\ x_2, t_2; \ldots;\ x_n, t_n)$$

In the theory of dynamical systems this description is based on <u>deterministic equations for the dynamics of probability distributions</u> $P(x_\xi(t) < x)$ or its densities $p(x,t)$.
(not for the instantaneous states of the system)

For problems with continuous time the densities are defined by an evolution operator

$$\frac{\partial p}{\partial t} = L p, \quad (2)$$

which is linear with respect to the densities. L contains the information on the system's behavior.

We will concentrate on the subclass of dynamical systems for which L "does not remember" the previous states (memoryless), but can depend on time (Markovian process).

(1) and (2) → these two definitions of stochastic processes determine two alternative approaches to the analysis of stochastic dynamical system

How to enclude fluctuations in the description of a system?

(1) add fluctuating sources into the dynamics and consider statistical ensemble

SDE or Langevin equation

(2) consider deterministic equations for the dynamics of probability densities

Fokker-Planck equation

Example: random movement of a particle in a fluid due to collisions with the molecules of the fluid.

Paul Langevin formulated a stoch. eq. of motion (SDE) for the time-dependent position $r(t)$ itself

A. Einstein formulated an evolution law for the probability $p(r,t)$ to find a particle in a certain position r at time t.

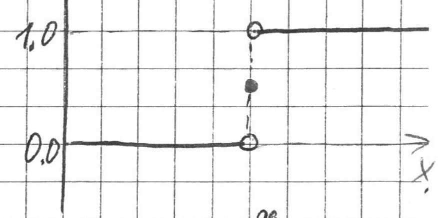
5.1. Probability distributions and densities

The probability distribution counts the number of realizations x_ξ at time t smaller than a fixed x:

$$P_x(x_\xi(t) < x) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \theta(x - x_{\xi_i}(t)) = \langle \theta(x - x_\xi(t)) \rangle \quad (3)$$

$\langle\ \rangle \to$ assigns the mean over $N \to \infty$ different realizations of $\xi(t)$

$\theta \to$ stands for the Heavyside function



$$\theta(x) = \int_{-\infty}^{x} \delta(s)\, ds$$

$$\theta(x) = \frac{d}{dx} \max\{x, 0\}$$

As a parametric function of x the distribution (3) varies between 0 and 1.

The probability density of the process $x_\xi(t)$ is $p(x, t) \to$ is defined by the amount of realizations that get into the interval $[x, x+dx)$ at time t:

$$p_x(x,t)\, dx = P_x(x \le x_\xi(t) < x + dx).$$

It can be obtained by taking the derivative of (3) with respect to x:

$$p_x(x,t) = \frac{d}{dx} P_x(x_\xi(t) < x) = \langle \delta(x - x_\xi(t)) \rangle, \quad (4)$$

where δ is Dirac's delta-function.

Properties of the probability density:

- normalization is given by $\int_X p_x(x,t)\, dx = 1$, where X is the set of possible values of x.
- if $y = f(x)$ one can present it as
$$y = \int f(x')\, \delta(x - x')\, dx' \text{; by averaging we get:}$$
$$\langle f(x) \rangle = \langle y(t) \rangle = \int f(x')\, p_x(x', t)\, dx',$$ which gives the mean of function $f(x)$.
- the probability density of y is
$$p_y(y,t) = \langle \delta(y - y_\xi(t)) \rangle = \langle \delta(y - f(x_\xi(t))) \rangle = \int_X \delta(y - f(x))\, p_x(x,t)\, dx \quad (5)$$

5.2 Joint and conditional probability

If t_i is an arbitrary number of subsequent times such that $i = 1, 2, \dots, n$, i.e., $t_0 < t_1 < t_2 < \dots < t_n$, then the complete information on the dynamics of the stochastic process $x(t)$ is given by the infinite sequence of n-time joint probability densities:

$$p_n(x_1, t_1; \dots; x_n, t_n) \overset{(4)}{=} \langle \delta(x_1 - x_\xi(t_1)) \dots \delta(x_n - x_\xi(t_n)) \rangle, \quad (6)$$

where $n = 2, 3, 4 \dots$

Another useful function (to characterise a stochastic process) is the conditional probability density:

$$p_n(x_n, t_n | x_1, t_1; \dots; x_{n-1}, t_{n-1}) = \langle \delta(x_n - x_\xi(t_n)) \rangle_{x_\xi(t_1) = x_1; \dots; x_\xi(t_{n-1}) = x_{n-1}} \quad (7)$$

It defines the probability density of the stochastic process x_ξ at time t_n with known history by holding x_ξ fixed at previous times t_i, $i = 1, \dots, n-1$.

$$(7) \Rightarrow p_n(x_1, t_1; \dots; x_n, t_n) = p_n(x_n, t_n | x_1, t_1; \dots; x_{n-1}, t_{n-1})\, p_{n-1}(x_1, t_1; \dots; x_{n-1}, t_{n-1}).$$

$n-1$ → defines how many moments in time we consider

5.3 Markovian processes

It is impossible to calculate a complete set of n-time joint probability densities. Fortunately, there is a special class of stochastic processes which allows a simplified description. These are Markovian processes: only the present state determines the future.

A process is called Markovian if the conditional probability density reduces to the transition probability density between ~~two~~ states at subsequent times independent of the states at previous times, i.e.,

$$p_n(x_n, t_n | x_1, t_1; \dots; x_{n-1}, t_{n-1}) = p_2(x_n, t_n | x_{n-1}, t_{n-1}) \quad (9)$$

For Markovian processes the transition probability density plays an important role since it holds that

$$p_n(x_1, t_1; \dots; x_n, t_n) = p_2(x_n, t_n | x_{n-1}, t_{n-1}) \dots p_2(x_2, t_2 | x_1, t_1)\, p(x_1, t_1) \quad (10)$$

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with $p(x_1, t_1)$ given by (4). Full information is given, therefore, by knowledge of the transition probability density and the probability density at the initial time.