An important relation between the transition probabilities can be formulated for ==Markovian processes.== Summing all possible densities at time t_2 by integrating p_3 over all possible states at time t_2 we get: (for the 3 states at three subsequent moments in time $t_1 < t_2 < t_3$)

joint prob.

$$p_2(x_1,t_1;x_3,t_3) = \int_X dx_2\, p_3(x_1,t_1;x_2,t_2;x_3,t_3) \quad (11)$$

and expanding the two joint probability densities (p_2 and p_3) at both sides according to (10) gives ==Chapman-Kolmogorov equation:==

$$\boxed{p_n(x_1,t_1;\dots x_n,t_n) = p_2(x_n,t_n|x_{n-1},t_{n-1})\dots \cdots p_2(x_2,t_2|x_1,t_1)\,p(x_1,t_1) \quad (10)}$$

cond. prob.

$$p_2(x_3,t_3|x_1,t_1) = \int_X dx_2\, p_2(x_3,t_3|x_2,t_2)\,p_2(x_2,t_2|x_1,t_1) \quad (12)$$

basic evolutionary equation for Markovian processes

One can rewrite (12) by distinguishing between tran-ns out of the given state and the prob-ty to stay in the state.

We introduce rates of jumps between different states, $\Delta x \neq 0$, by (freq-ies) (x and Δx) at time t

$$W(x \to x+\Delta x, t) = \lim_{dt\to 0} \frac{1}{dt} p_2(x+\Delta x, t+dt|x,t) \quad (13)$$

(time interval is small)

The full transition probability may be replaced for <u>short</u> times by

$$p_2(x,t+dt|x+\Delta x,t) = \left(1 - dt\int dx'\,W(x+\Delta x \to x',t)\right)\delta(\Delta x) + W(x+\Delta x \to x,t)\,dt + O(dt^2) \quad (14)$$

The first term is the probability remaining in state x with $\Delta x = 0$.

Substituting (14) into the integral of (12) by appropriate assignments gives for the limit $dt \to 0$

$$\frac{\partial}{\partial t} p_2(x,t|x_1,t_1) = \int dx' \Big[\overset{\text{"in"}}{W(x'\to x,t)\,p_2(x',t|x_1,t_1)} - \overset{\text{"out"}}{W(x\to x',t)} \cdot p_2(x,t|x_1,t_1)\Big] \quad (15)$$

(15) is called the master equation, or Pauli equation in the case of discrete events. It expresses the temporal evolution of probability ~~at~~ characteristics state x through the balance of out- and inflowing probability.

5.4. Stationarity

A stochastic process is called strictly stationary if the joint probability density depends on the difference τ between two subsequent times. A change of $t_i \to t_i + \tau$ for all $i = 1, 2, \ldots, n$ does not affect the probability density. One may also state that the process is independent ~~of~~ the initial time t_0.

As conclusion one gets immediately for the transition probability density:

$$p_2(x_2, t+\tau / x_1, t) = p_2(x_2, \tau / x_1) \quad (16)$$

It depends on the time $\tau = t_2 - t_1$ difference and is, therefore, time homogeneous.

An important measure characterizing stationarity of a process is its limit final stationary probability density. If ergodic, the process loses its dependence on the initial state x_0. The transition probability density in the limit $\tau \to \infty$ approaches the stationary probability density:

$$p^s(x) = \lim_{\tau \to \infty} p_2(x, \tau / x_0) \quad (17)$$

Asymptotically periodic stochastic processes possess/have asymptotic probability densities which vary periodically in time:

$$p(x, t - t_0) = p(x, t + T - t_0).$$

These processes are invariant with respect to time shifts of one period. Example: periodically driven stochastic systems.

5.5. Moments of stochastic processes

determ. functions of t or constants

Stochastic processes can be defined by their time-dependent moments. The number of moments which is sufficient to define the stochastic process $\xrightarrow{\text{gives}}$ the number of independent parameters of the process.

Gaussian processes
↓
can be characterized through the first and second moment only (all higher cumulants vanish)

Exponential or Poissonian processes
↓
all higher moments are expressed through the first moment.

The conditional first moment of the transition probability density is given by

$$\langle x_\xi(t)\rangle_{x_\xi(t_0)=x_0} = \int dx\, x\, p_2(x,t\,|\,x_0,t_0) \quad (18)$$

In the asymptotic limit $t_0 \to -\infty$, it loses its dependence on the initial time:

$$\langle x_\xi(t)\rangle_{asy} = \lim_{t_0\to-\infty} \langle x_\xi(t)\rangle_{x_\xi(t_0)=x_0}, \quad (19)$$

if the transition probability density converges in this limit to $p(x,t)$:
$p_2(x,t\,|\,x_0,t_0) \to p(x,t)$.

For stationary processes this limit is independent of time, and, therefore, without loss of generality, we can assume
$\langle x\rangle_{asy} = 0$.

An important characterization is given by the second moment defined at two different times. Again we have an initial state condition: $x_\xi(t_0) = x_0$

$$\langle x_\xi(t)\, x_\xi(t+\tau)\rangle_{x_\xi(t_0)=x_0} = \int dx_1\, dx_2\, x_1 x_2\, p_2(x_2, t+\tau \,|\, x_1, t) \cdot p_2(x_1, t \,|\, x_0, t_0) \quad (20)$$

It is known as the autocorrelation function of the stochastic process x. It may be expressed through the conditional first moment:

$$\int dx_2 \langle x_\xi(t+\tau)\rangle_{x_\xi(t)=x_1}\, x_1\, p_2(x_1, t \,|\, x_0, t_0) \quad (21)$$

If $t \to -\infty$ the autocorrelation function (ACF) approaches its asymptotic limit:

$$\langle x_\xi(t)\, x_\xi(t+\tau)\rangle_{asy} = \int dx_1\, dx_2\, x_1 x_2\, p_2(x_1, t, x_2, t+\tau) \quad (22)$$

For $\tau = 0$ it equals to the second moment $\langle x^2(t)\rangle$. On the other hand, if $\tau \to \infty$ (22) factorizes in many cases into:

$$\langle x_\xi(t)\, x_\xi(t+\tau)\rangle_{asy} = \langle x_\xi(t+\tau)\rangle_{asy} \langle x_\xi(t)\rangle_{asy} \quad (23)$$

For stationary processes the asymptotic values do not depend explicitly on time, and the stationary ACF remains a function of the time difference τ only:

$$C_{x,x}(\tau) = \int dx_1\, dx_2\, x_1 x_2\, p_2(x_1, \tau \,|\, x_2)\, p(x_2). \quad (24)$$

The stationary process is invariant with respect to the shifts in time $t \to t-\tau$, and it follows that C_{xx} is an even function in time, $C_{x,x}(\tau) = C_{x,x}(-\tau)$.

The ACF gives an important measure of the stochastic process → the correlation time τ_c. It can be defined in several ways, but most often it is defined as

$$\tau_c = \frac{1}{C_{x,x}(0)} \int_0^\infty dt \, |C_{x,x}(t)| \quad (25)$$

Example: two-dimensional stochastic process $\{x(t), y(t)\}$.

Cross-correlations can be similarly introduced on the basis of the joint probability density $p_{x,y}(x,t; y, t+\tau)$. For the stationary process it gives: $C_{x,y}(\tau) = C_{y,x}(-\tau)$. The absolute value of the correlation function obeys:

$$|C_{x,y}(\tau)|^2 \leq \langle x^2 \rangle \langle y^2 \rangle$$

Cauchy-Schwartz inequality

Therefore, for the ACF:

$$|C_{x,x}| \leq \langle x^2 \rangle$$