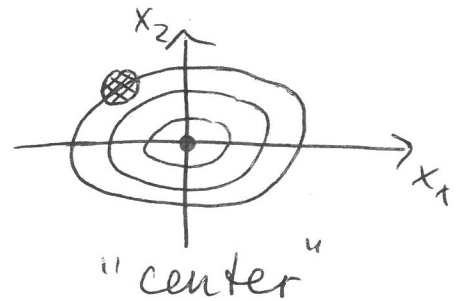


Phase portrait $\rightarrow$ phase space with phase trajectories

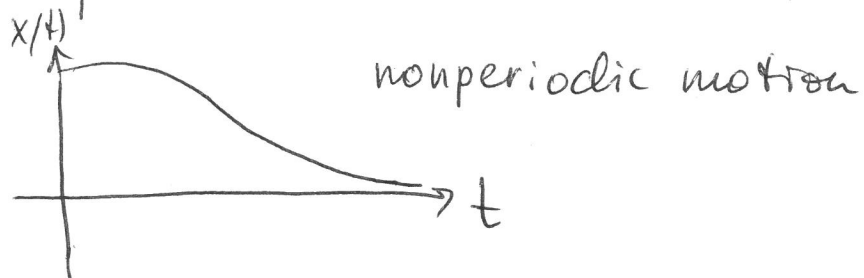
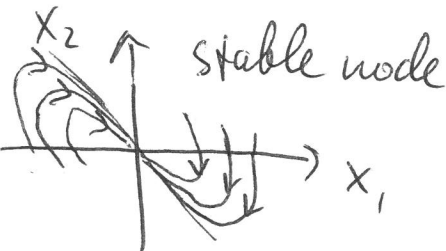
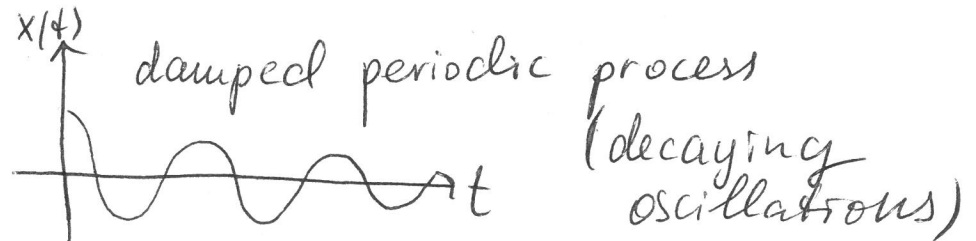
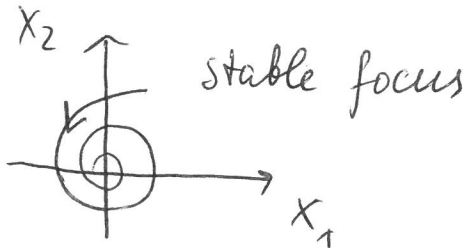
periodicity

- closed trajectories
- phase volume $\Delta V = \text{const}$



dissipative systems

- for $t \rightarrow \infty$ $\Delta V \rightarrow 0$



Dyn. systems (classification)

- continuous (flow)
- discrete-time (maps)
- linear
- nonlinear

• autonomous $\vec{x} = \vec{F}(\vec{x}, \vec{\mu})$ $\xrightarrow{\text{parameters}}$

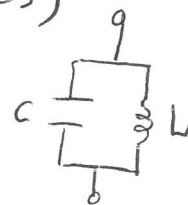
(the evolution law does not explicitly depend on time; no external forcing)

• nonautonomous $\vec{x} = \vec{F}(\vec{x}, \vec{\mu}) + \vec{g}(t)$

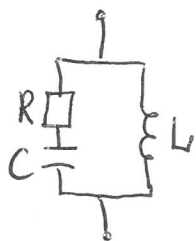
noise or periodic external forcing

- conservative (Hamiltonian in mechanics)

(2)



- dissipative



(energy decreases in time due to friction or dissipation)

phase volume element is time-dependent

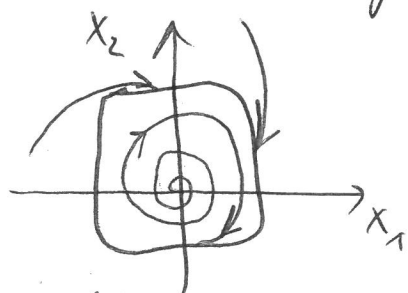
special case

systems with negative dissipation
(ΔV increases)

systems with positive dissipation
(ΔV decreases)

Self-sustained oscillations : nonlinear and nonconservative
the basic characteristics (amplitude, frequency, waveform)
are determined by system parameters and are independent, within certain limits, of initial conditions

Limit cycle : an isolated closed trajectory in the phase space
in the vicinity of the limit cycle there are no more closed orbits!



stable limit cycle

Examples:

- beating of a heart
- daily rhythms in human body temperature and hormone secretion
- self-excited in bridges and airplane wings

Fixed points (steady states) is a state at which the system does not change its behaviour in time.

$$\begin{cases} f_1(x_1, \dots, x_N) = 0 \\ \vdots \\ f_N(x_1, \dots, x_N) = 0 \end{cases}$$

(no evolution in time $\rightarrow$ derivative = 0)

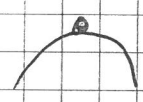
stable

small disturbances
— damp out in time



unstable

small disturbances away from the steady state grow with time



Linear stability analysis

$N=2$

1)
$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$$

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases} \Rightarrow (x_0, y_0)$$

$\tilde{x}, \tilde{y} \rightarrow$ small disturbances

(2)
$$\begin{aligned} x &= x_0 + \tilde{x} \\ y &= y_0 + \tilde{y} \end{aligned}$$

We substitute (2) into (1) and apply Taylor's expansion to g and f in the vicinity of (x_0, y_0) .

We neglect square terms and all other terms with order larger than 2.

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= f(x_0, y_0) + \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} \tilde{x} + \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} \tilde{y} \\ \frac{d\tilde{y}}{dt} &= g(x_0, y_0) + \frac{\partial g}{\partial x} \Big|_{(x_0, y_0)} \tilde{x} + \frac{\partial g}{\partial y} \Big|_{(x_0, y_0)} \tilde{y} \end{aligned}$$

This are the equations for the evolution of the small disturbances in the vicinity of the fixed point

Taylor series
Taylor's expansion

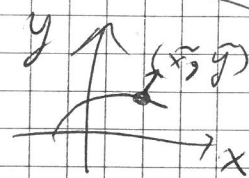
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k =$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x} + B\tilde{y}$$

$$\frac{d\tilde{y}}{dt} = C\tilde{x} + D\tilde{y}$$

$$\frac{d}{dt} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$



$$\tilde{x} = \tilde{x}_1 e^{\lambda_1 t}$$

$$\tilde{y} = \tilde{x}_2 e^{\lambda_2 t}$$

We substitute the solution into the linear equations

$$\begin{cases} (A - \lambda) \tilde{x}_1 + B \tilde{x}_2 = 0 \\ C \tilde{x}_1 + (D - \lambda) \tilde{x}_2 = 0 \end{cases}$$

In order to have solutions (the condition) the determinant should be zero!

Jacobian matrix

$$\begin{vmatrix} A - \lambda & B \\ C & D - \lambda \end{vmatrix} = 0$$

characteristic equation

$$\lambda^2 - (A+D)\lambda + AD - BC = 0$$

$$\underbrace{A+D}_{\text{trace of a matrix}} = S \text{ ("Spur")}$$

$$\underbrace{AD - BC}_{\text{determinant of the matrix}} = J$$

$$\boxed{\lambda^2 - S\lambda + J = 0}$$

$$\lambda_{1,2} = \frac{S}{2} \pm \sqrt{\frac{S^2}{4} - J}$$

$N=2 \Rightarrow$ phase space dimension is 2 $\Rightarrow$ 2 eigenvalues
 $\tilde{x} \sim e^{\lambda_1 t}$, $\tilde{y} \sim e^{\lambda_2 t}$

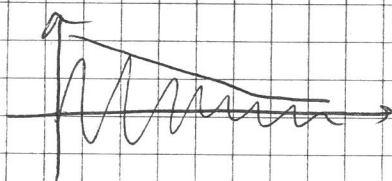
$\lambda_{1,2}$ - real and negative $\Rightarrow$ stable fixed point (5)

$\lambda_{1,2}$ - complex $\rightarrow \lambda = d + i\omega$

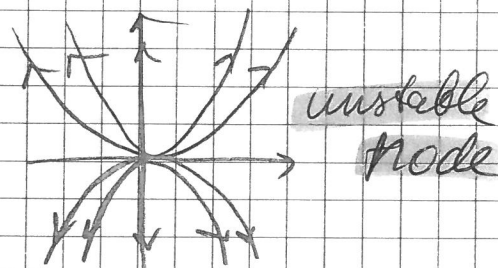
and negative real parts

$\Rightarrow$ stable fixed point (focus)

$$\tilde{x} \sim e^{\alpha t} \underbrace{e^{j\omega t}}_{\text{rotation}}$$



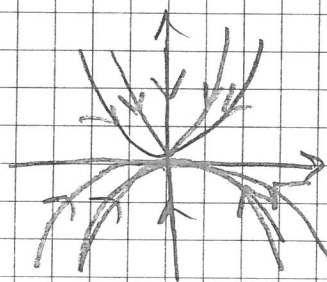
1) λ_1, λ_2 - real and positive
 $\lambda_1 > 0$
 $\lambda_2 > 0$



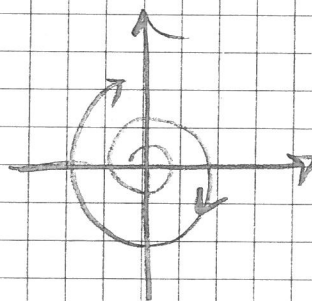
2) λ_1, λ_2 - real
 $\lambda_1 > 0$
 $\lambda_2 < 0$



3) λ_1, λ_2 real
 $\lambda_1 < 0$
 $\lambda_2 < 0$



4) λ_1, λ_2 - complex
 $\text{Re } \lambda_1, \lambda_2 > 0$

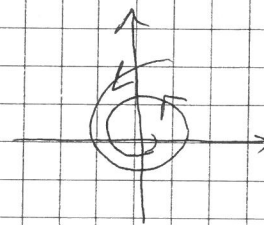


6

5) λ_1, λ_2 - complex

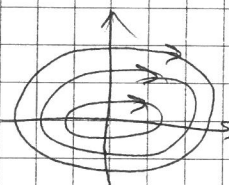
$$\operatorname{Re} \lambda_1 \lambda_2 < 0$$

stable focus



6) λ_1, λ_2 - imaginary

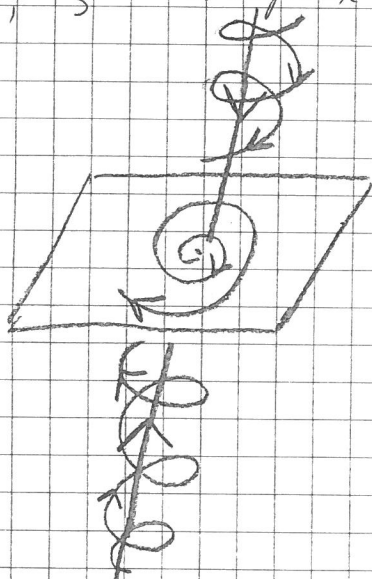
center



$N=3$ saddle-focus

λ_1 - real and $\lambda_1 < 0$

λ_2, λ_3 - complex, $\operatorname{Re} \lambda_{2,3} > 0$



λ_1 - real and $\lambda_1 > 0$

λ_2, λ_3 complex

$$\operatorname{Re} \lambda_{2,3} < 0$$

