5.6

Wiener process $W(t)$ – continuous-time stochastic process named in honor of Norbert Wiener – Am. math. (1894 – 1964)

Brownian motion can be described as a Gaussian process.

W_i – position of the Brownian particle at time t_i.

Joint distribution for Wiener process:

$$p_n(W_1, t_1; \dots, W_n, t_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{4\pi D(t_i - t_{i-1})}} \cdot \exp\left(-\frac{1}{4D}\frac{(W_i - W_{i-1})^2}{t_i - t_{i-1}}\right), \qquad (26)$$

where $W_0 = 0$, $t_0 = 0$

– It is Markovian since the factors with $i \geq 2$ in the product (26) are the transition probabilities between two subsequent positions:

$$\boxed{p(W_i, t_i | W_{i-1}, t_{i-1}) = \frac{1}{\sqrt{4\pi D(t_i - t_{i-1})}} \exp\left(-\frac{(W_i - W_{i-1})^2}{4D(t_i - t_{i-1})}\right)} \qquad (27)$$

They do not depend on the former history and satisfy Chapman-Kolmogorov equation (12).

– This process is nonstationary. With the initial condition $p(W_0, t_0 = 0) = \delta(W_0)$, one finds for $t \geq 0$ the probability density:

$$p_1(W, t) = \int dW_0\, p(W, t | W_0, t_0)\, \delta(W_0) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{W^2}{4Dt}\right). \qquad (28)$$

One can see that the prob. density explicitly depends on time.

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

The first moment of the Wiener process vanishes for the given I.C. $\langle W(t)\rangle = 0$, but the second moment increases linearly in time:

$$\langle W(t_1)\, W(t_2)\rangle = 2\mathcal{D} \min(t_1, t_2) \qquad (29)$$

For $t_1 = t_2 = t$: $\langle W(t)^2\rangle = 2\mathcal{D}t \Rightarrow$ the parameter $\mathcal{D}$ defines the rate of the variance growth. In the case of Brownian motion this rate is just the diffusion coefficient.

Variance

$Var(X) = \langle (X - \langle X\rangle)^2\rangle =$

$= \langle X^2\rangle - \langle X\rangle^2$

expectation of the squared deviation of a random variable from its mean

An important conclusion can be made on the $\boxed{\text{increments}}$ (Zuwachs, Zunahme) of the Wiener process:

$$\Delta W_{t_1,t_2} = W(t_2) - W(t_1);\ t_2 > t_1$$

The increments are Gaussian-distributed due to (27) (trans. prob-ies) with zero mean: $\langle \Delta W_{t_1,t_2}\rangle = 0 \quad (30)$

The increments satisfy the stationarity condition:

$$\langle (\Delta W_{t_1,t_2})^2\rangle = 2\mathcal{D}(t_2 - t_1) \qquad (31)$$

(depends only on time difference)

A very important property of the Wiener process is the independence of the increments for the nonoverlapping (disjoint) intervals of time.

$$\langle \Delta W_{t_1,t_2}\, \Delta W_{t_2,t_3}\rangle = 0\ ; \quad t_3 > t_2 > t_1 \qquad (32)$$

Subsequent increments of the Wiener process form a stationary Markovian process, since (27) is an exact solution of the Chapman-Kolmogorov equation (trans. prob-ies) $\Rightarrow \Delta W_{t_1,t_2} = \Delta W_{t_2-t_1}$

In its differential form for $t_{i+1} - t_i \to 0$, this equation may be presented as a diffusive one:

$$\frac{\partial}{\partial t} p(\Delta W, t / \Delta W_0, t_0) = \mathcal{D} \frac{\partial^2}{\partial \Delta W^2} p(\Delta W, t / \Delta W_0, t_0) \qquad (33)$$

Wiener process – main properties

1) Gaussian process, Markovian process
2) Its mean is defined by the I.C.: $\langle W(t)\rangle = W_0$
3) Nonstationary, the variance grows linearly in time:

$$\sigma_W^2(t) = 2D(t-t_0)$$

! 4) Wiener process is a process with <u>independent increments</u> for the nonoverlapping intervals of time

6. The Fokker-Planck equation

6.1. Differential form of the Chapman-Kolmogorov equation

An approach to describe Markovian processes is based on evolution equations for the transition probability density $p(x,t/x_0,t_0)$, which guarantees full knowledge of Markovian dynamics. Let us consider differential operators for this transition probability density.

Generally, there are two different evolution problems, as $p(x,t/x_0,t_0)$ depends on two pairs of x,t and x_0,t_0 giving adjoint evolution operators for the transition probability density.

"forward" problem

to study the evolution of prob. density varying the state x at time $t>t_0$ for the fixed initial state x_0 at time t_0.

"backward" problem

to consider the initial state x_0 and time t_0 as variables for the fixed x, t.

The type of the problem to consider depends on the particular situation: a "backward" problem arises, for example, when studying the time difference $t-t_0$ which trajectories of a stochastic process require to reach a fixed value x from some initial point x_0.

The derivation of both differential operators starts with the Chapman-Kolmogorov equation and can be found in many books on stoch. processes (for example, Gardiner, "Handbook of Stochastic methods." Springer Series in Synergetics. Vol. 13. Berlin, Heidelberg: Springer, 1982).

The forward differential form is:

$$\frac{\partial p(x,t/x_0,t_0)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n} K_n(x,t)\, p(x,t/x_0,t_0). \quad (34)$$

$K_n(x,t)$ are kinetic coefficients and are defined as rates of evolution for conditional averges:

$$K_n(x,t) = \frac{1}{n!} \lim_{dt\to 0} \frac{1}{dt} \int [x-x']^n p(x',t+dt/x,t)\, dx' = \quad (35)$$

$$= \frac{1}{n!} \lim_{dt\to 0} \frac{\langle dx^n \rangle}{dt} \Big|_{x(t)=x}$$

(34) is Kramers-Moyal expansion of the Chapman-Kolmogorov equation and is valid for any Markovian process with given coefficients (35).

Kramers-Moyal expansion for the backward problem has the form:

$$-\frac{\partial p(x,t/x_0,t_0)}{\partial t_0} = \sum_{n=1}^{\infty} K_n(x_0,t_0) \frac{\partial^n p(x,t/x_0,t_0)}{\partial x_0^n} \quad (36)$$

The evolution operators in (34) and (36) are adjoint to each other and can be simplified significantly for so-called diffusion processes.

For these processes the only two coefficients (35) are nonzero ($n=1$, $n=2$), while $K_n = 0$ for $n > 2$. In this case (34) and (36) are called forward and backward FPEs, respectively.

Therefore, forward FPE is

$$\frac{\partial p(x,t/x_0,t_0)}{\partial t} = L_x^F \, p(x,t/x_0,t_0) =$$

$$= -\frac{\partial}{\partial x} K_1(x,t) p(x,t/x_0,t_0) + \frac{\partial^2}{\partial x^2} K_2(x,t) p(x,t/x_0 t_0) \quad (37)$$

And the backward version is:

$$-\frac{\partial p(x,t/x_0,t_0)}{\partial t_0} = L_{x_0}^B \, p(x,t/x_0,t_0) =$$

$$= K_1(x_0,t_0) \frac{\partial p(x,t/x_0,t_0)}{\partial x_0} + K_2(x_0,t_0) \frac{\partial^2 p(x,t/x_0,t_0)}{\partial x_0^2} \quad (38)$$

Generalization to the case of many dynamical variables $x_1, x_2, \dots, x_n$ is straightforward.

For $p = p(x_1,\dots,x_n,t/x_{10},\dots,x_{n0},t_0)$ and corresponding conditioned averages one can get:

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} K_1^i(x_1,\dots,x_n,t) p + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} K_2^{i,j}(x_1,\dots,x_n,t) p; \quad (39)$$

where

$$K_1^i(x_1,\dots x_n,t) = \lim_{dt\to 0} \frac{\langle dx_i \rangle}{dt} \Big|_{x_1(t)=x_1,\dots,x_n(t)=x_n}$$

$$K_2^{i,j}(x_1,\dots x_n,t) = \frac{1}{2} \lim_{dt\to 0} \frac{\langle dx_i dx_j \rangle}{dt} \Big|_{x_1(t)=x_1,\dots,x_n(t)=x_n}.$$