English Summary

2 Phenomenological Models

+ few equations, simple nonlinearities, feasible for bifurcation analysis
- only qualitative agreement of time series, low physiological correspondence

2.1 FitzHugh-Nagumo model

$$\varepsilon \dot{x} = x - \frac{x^3}{3} - y \qquad x: \text{activator} \qquad a: \text{bifurcation parameter}$$

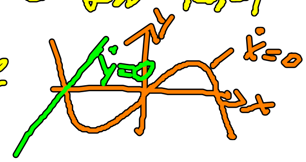
$$\dot{y} = x + a - \gamma y \qquad y: \text{inhibitor} \qquad \varepsilon: \text{time scale separation} \quad \varepsilon \ll 1 \;\begin{cases} x: \text{fast} \\ y: \text{slow} \end{cases}$$

extended FHN

$|a|<1$: oscillatory (limit cycle)
$|a|>1$: excitable (fixed point)

$(x^*, y^*) = \left(-a, -a + \frac{a^3}{3}\right)$ unstable for $|a|<1$, stable for $|a|>1$

- also known / initially introduced as Bonhoeffer-van der Pol model



- Canard trajectory separates subthreshold oscillations from large excursions
- in phase space
- excitability type II: at bifurcation point, limit cycle starts with amplitude 0 and finite frequency $\operatorname{Im} \lambda \big|_{|a|=1} = \frac{1}{\sqrt{\varepsilon}}$

2.2 SNIPER model

Saddle-node infinite period bifurcation

$$\dot{x} = x(1 - x^2 - y^2) + y(x - b)$$
$$\dot{y} = y(1 - x^2 - y^2) - x(x - b)$$

$\overset{x = r\cos\varphi,\; y = r\sin\varphi}{\Longleftrightarrow}$

$$\dot{r} = r(1 - r^2)$$
$$\dot{\varphi} = b - r\cos\varphi$$

fixed points: trivial $(x_A^*, y_A^*) = (0,0)$ with eigenvalues $\lambda_{1,2} = 1 \pm ib$ (focus)

exist only for $|b|<1$:
$(x_B^*, y_B^*) = (+b, \sqrt{1-b^2})$ with $\lambda_1 = -2$, $\lambda_2 = \sqrt{1-b^2}$ (saddle)
$(x_C^*, y_C^*) = (+b, -\sqrt{1-b^2})$

λ_1: stable, λ_2: unstable direction

2.2 SNIPER-Modell (Fortsetzung)

Jacobi-Matrix:

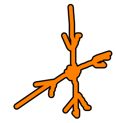
$$\underline{\underline{DF}} = \begin{pmatrix} 1 - 3x^2 - y^2 + y & -2xy + x - b \\ -2xy - 2x - b & 1 - x^2 - 3y^2 \end{pmatrix}$$

3. Fall: $(x_C^*, y_C^*) = (b, -\sqrt{1-b^2}) \Rightarrow$ charakteristische Gleichung

$$\det\left(\underline{\underline{DF}}\big|_{x_C^*, y_C^*} - \lambda \mathbb{1}\right) = \ldots = \underbrace{(\lambda+2)}\;\underbrace{(\lambda + \sqrt{1-b^2})}$$

$\Lambda_1 = -2 \quad \Lambda_2 = -\sqrt{1-b^2}$

$\Rightarrow$ beide Eigenwerte sind negativ $\Rightarrow$ Knoten (node)
(zwei stabile Richtungen)



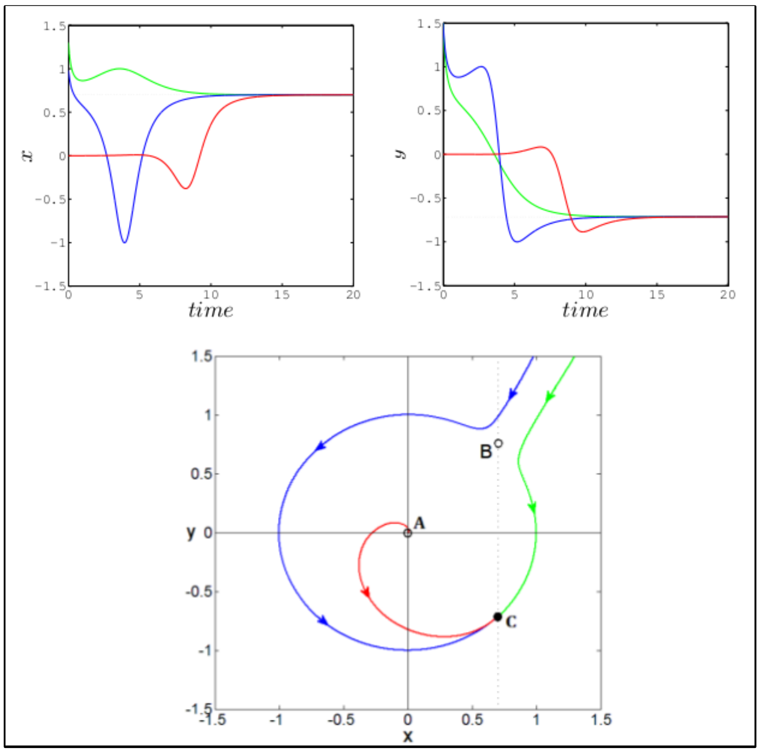
An der Bifurkation $|b| = 1$: $(x_B^*, y_B^*) = (1, 0) = (x_C^*, y_C^*)$

$\Rightarrow$ Fixpunkte B und C entstehen gleichzeitig bei $|b| = 1$

$\Rightarrow$ intrinsische Zeitskala $T \sim \frac{1}{\mathrm{Im}\,\Lambda} \to \infty$ (infinite period)

Dynamische Szenarien:

(i) unterhalb der Bifurkation $|b| < 1$ z.B.: $b = 0.7$



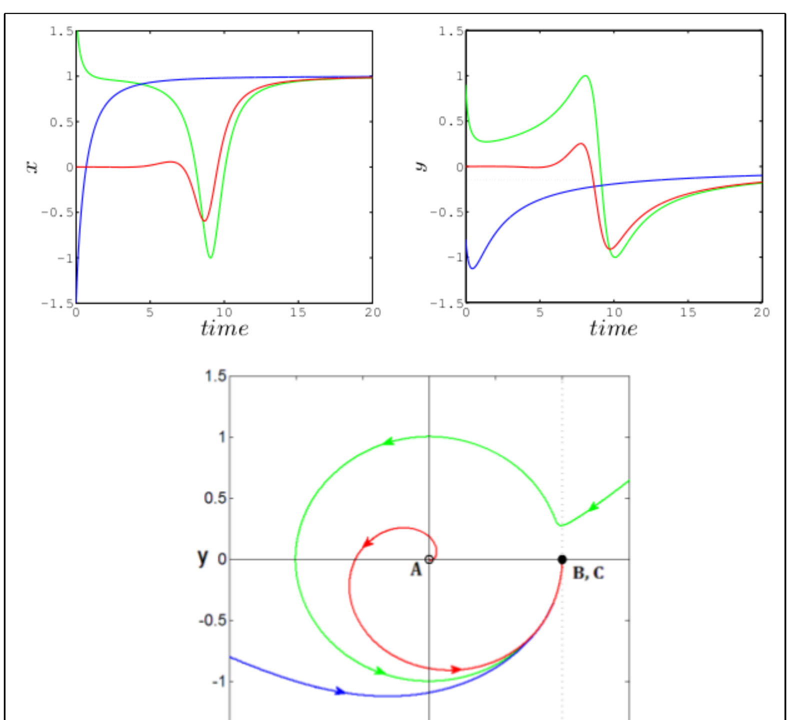
3 Fixpunkte:

Fokus A

Sattel B

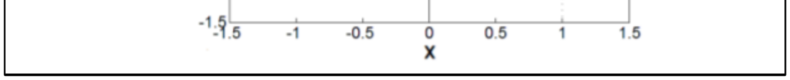
stabiler Knoten C

(ii) an der Bifurkation $|b| = 1$

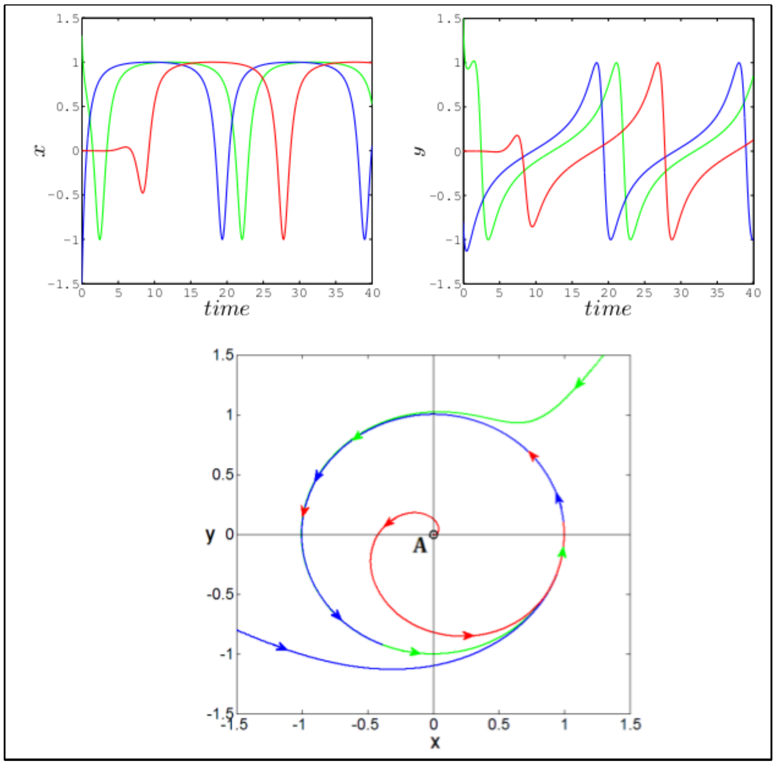


Fokus A

Fixpunkte (Sattel & Knoten) kollidieren



(iii) oberhalb der Bifurkation $|b|>1$ z.B. $b=1.05$



Fokus A

Grenzzyklus mit Radius $\sqrt{x^2+y^2}=1$

langsame Dynamik in der Nähe der ehemaligen Fixpunkte B, C, d.h. in der Nähe von $(1,0)$

$\Rightarrow$ Geist der Sattelknoten-Bifurkation

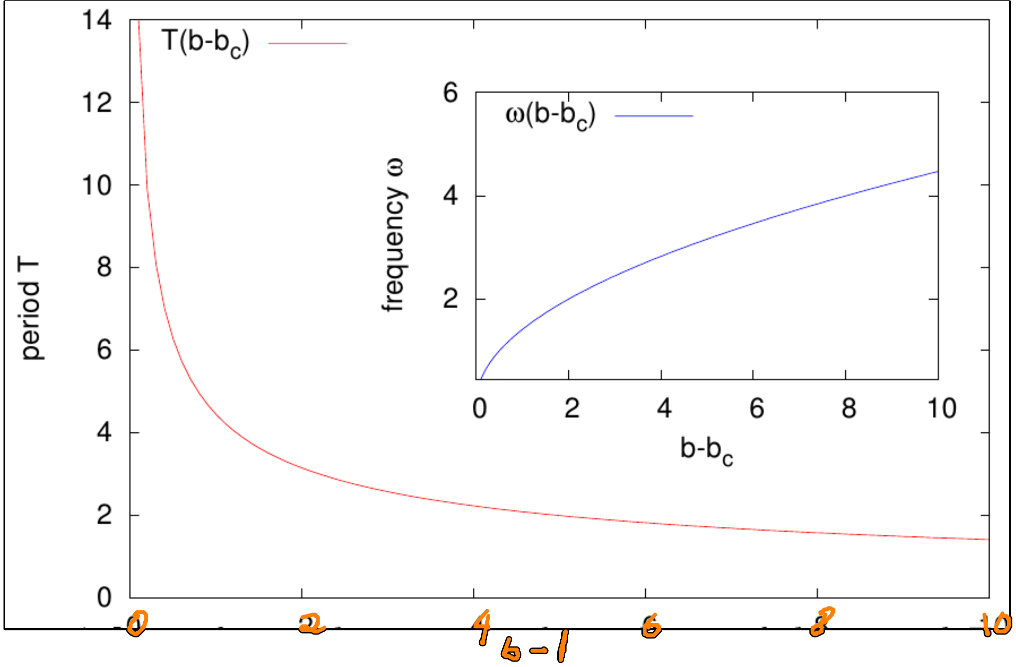
Berechnung der Periode des Grenzzyklus $(|b|>1)$:

$$\dot{\varphi} = b - \underset{r=1}{\cos\varphi} \quad \Rightarrow \quad d\varphi = (b-\cos\varphi)\,dt$$

$$\Rightarrow \quad \frac{d\varphi}{b-\cos\varphi} = dt$$

Periode $T = \int_0^{2\pi} \frac{d\varphi}{b-\cos\varphi} = \dots = \frac{2\pi}{\sqrt{b^2-1}}$

$\Rightarrow$ Periode T divergiert für $|b| \to 1$



unendliche Periode T
bei endlicher Amplitude
r=1
⇒ Anregbarkeit Typ I

2.3 Hindmarsh-Rose-Modell

zweidimensionale Version

Parameter

$$\dot{x} = c\left(x - \frac{x^3}{3} - y + z\right)$$

$$\dot{y} = \frac{1}{c}\left(x^2 + dx - by + a\right)$$

kubische Nullkline für x

quadratische Nullkline für y

7. Egan, R. W., Galen, P. H., Beveridge, G. C., Phillips, G. B. & Marnett, L. J. *Prostaglandins* **16,** 861–869 (1978).
8. Mason, R. T. & Staszewska-Barczak, J. *J. clin. exp. Pharmac. Physiol.* **6,** 678–685 (1979).
9. Takeguchi, C., Kohno, E. & Sih, C. J. *Biochemistry* **10,** 2372–2376 (1971).
10. Winter, C. A., Risley, E. A. & Nuss, G. W. *Proc. Soc. exp. Biol. Med.* **111,** 554–557 (1962).
11. Branceni, D., Azadian-Boulanger, G. & Jequier, R. *Archs int. Pharmacodyn.* **152,** 15–24 (1964).
12. Kemper, F. & Ameln, G. *Z. ges. exp. Med.* **131,** 407–415 (1959).
13. Flückiger, E., Schlach, W. & Taeschler, M. *Schweiz. med. Wschr.* **93,** 1232–1237 (1963).

A model of the nerve impulse using two first-order differential equations

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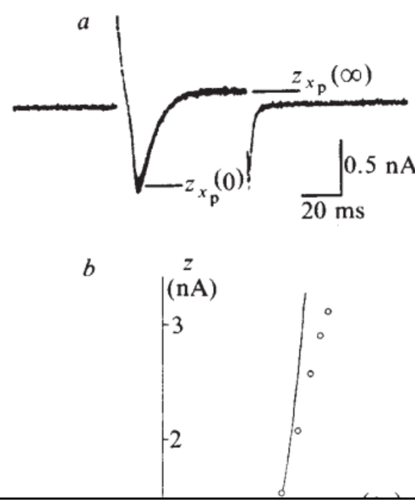

Thus the assumed form for our equations is

$$\dot{x} = -a(f(x) - y - z) \tag{7}$$

$$\dot{y} = b(f(x) - q\,e^{rx} + s - y) \tag{8}$$

where $f(x) = cx^3 + dx^2 + ex + h$, and a–h, q, r and s are constants.
After measuring a and b, and fitting cubic and exponential functions to the $z_{x_p}(0)$ and $z_{x_p}(\infty)$ data of Fig. 1b, the solutions of equations (7) and (8) were obtained by numerical integration.

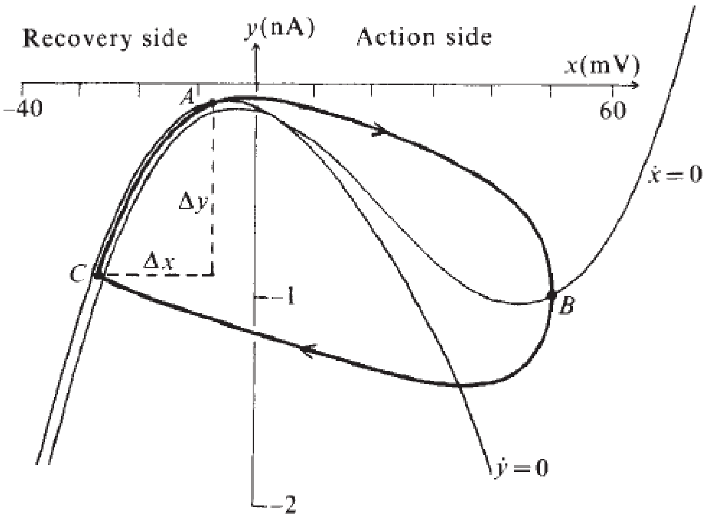

Fig. 3 Phase plane representation of the limit cycle solution to equations (7) and (8). The values of the constants are the same

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BIFURCATIONS IN TWO-DIMENSIONAL HINDMARSH–ROSE TYPE MODEL

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Bestimmung der Fixpunkte: $\dot{x} = 0$, $\dot{y} = 0$

$\Rightarrow$ Lösung der Gleichung: $\alpha x^3 + \beta x^2 + \gamma x + \delta = 0$

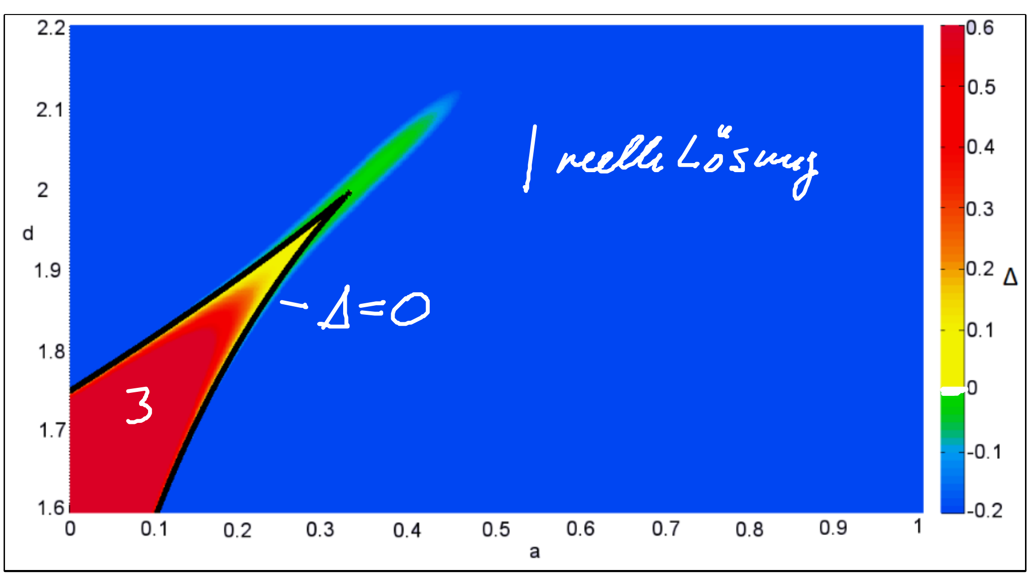
z.B.: $b=1$, $c=3$, $z=0$: $\alpha = 1$, $\beta = 3$, $\gamma = 3(d-1) + 3a$

$\Rightarrow$ Berechne die sog. Diskriminante $\Delta = \beta^2\gamma^2 - 4\alpha\gamma^3 - 4\beta^3\delta - 27\alpha^2\delta^2 + 18\alpha\beta\gamma\delta$

(a) $\Delta > 0$: 3 reelle Lösungen

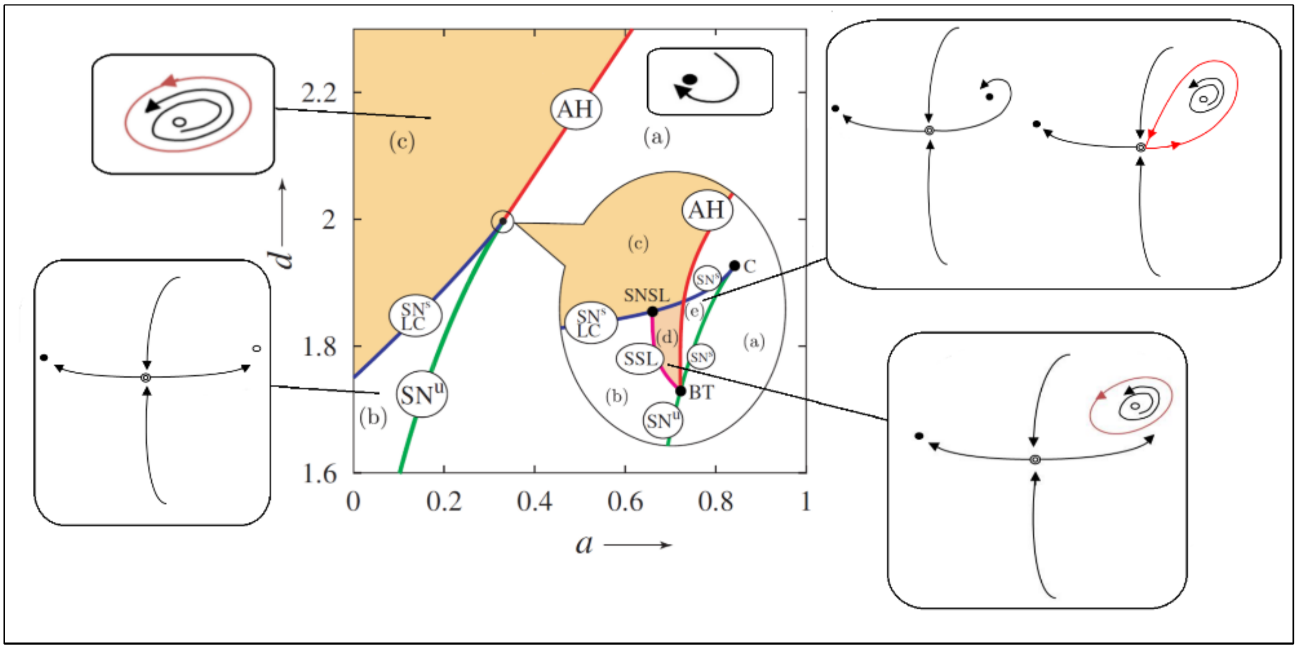
(b) $\Delta = 0$: 1 Lösung mit Vielfachheit 2, 1 weitere Lösung

(c) $\Delta < 0$: 1 reelle Lösung, 2 komplex konjugierte Lösungen



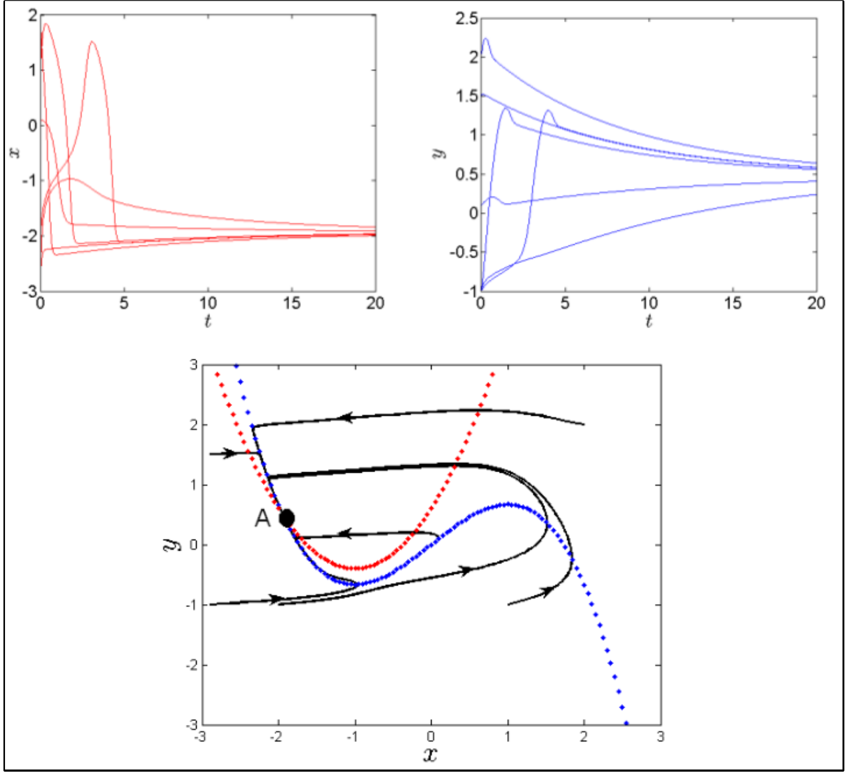
Eigenwerte per linearer Stabilitätsanalyse:

$$DF = \begin{pmatrix} c - cx^2 & -c \\ \frac{1}{c}(2x+d) & -\frac{b}{c} \end{pmatrix}$$



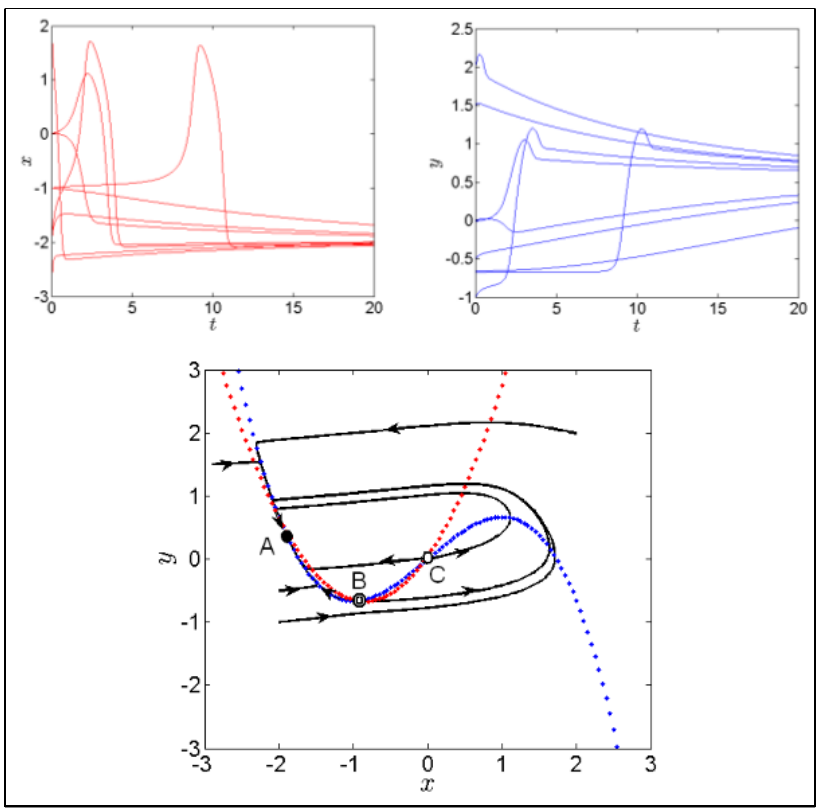
Label	Description
(a)	1 stable fixed point
(b)	1 saddle, 1 stable, 1 unstable fixed point
(c)	1 unstable fixed point, 1 stable limit cycle
(d)	1 saddle, 1 stable fixed points, 1 unstable fixed point, 1 stable limit cycle
(e)	1 saddle, 2 stable fixed points
AH	(Andronov-)Hopf bifurcation
SNLC	Saddle-node bifurcation on a limit cycle
SN	Saddle-node bifurcation (of equilibria)
BT	Bogdanov-Takens bifurcation
C	Cusp bifurcation
SSN	Saddle-separatrix loop bifurcation
SNSL	Saddle-node on separatrix loop bifurcation

(a) $a=0.6$, $d=1.7$



stabiler Knoten $\lambda_{1,2} < 0$

(b) $a = 0.05$, $d = 1.7$



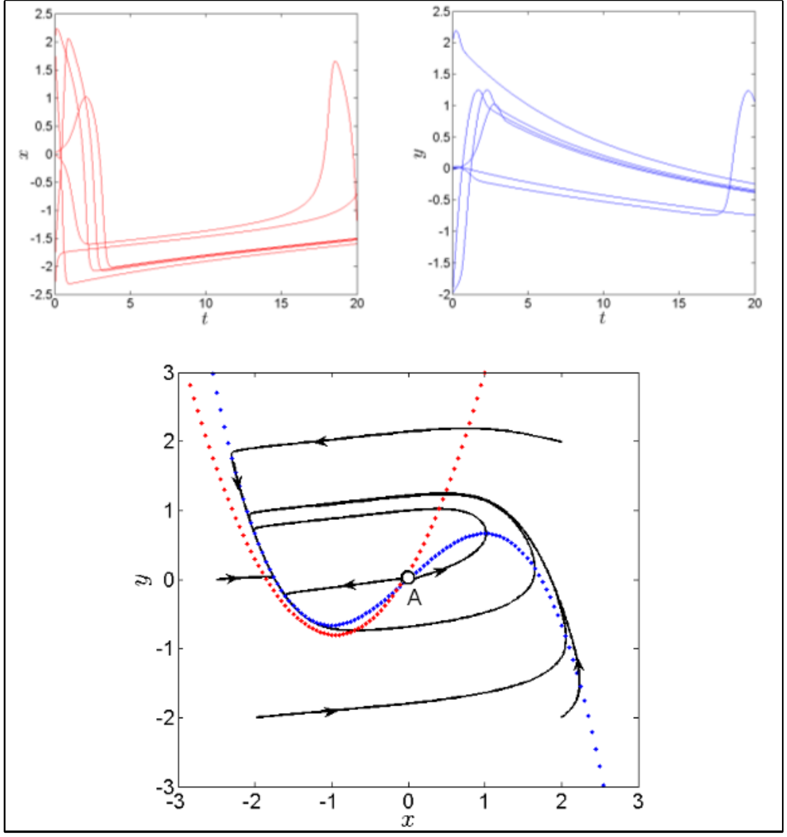
3 Fixpunkte:

stabiler Knoten A: $\lambda_{1,2} < 0$

instabiler Knoten B: $\lambda_{1,2} > 0$

Sattelpunkt C : $\lambda_1 < 0$, $\lambda_2 > 0$

(c) $a = 0.1$, $d = 1.3$



instabiler Knoten $\lambda_{1,2} > 0$

Hindmarsh-Rose-Modell mit 3 Variablen:

$\dot{x} = y - a x^3 + b x^2 - z$ kubisch

$\dot{y} = c - d x^2 - y$ quadratisch

$\dot{z} = \varepsilon (s (x - x_0) - z)$ linear

→ Bursting-Verhalten