English summary:

1.1 Dynamical systems

described as system of (ordinary) differential equations: $\dot{\underline{x}}(t) = \underline{F}(\underline{x}(t), t)$

$\underline{x} \in \mathbb{R}^n$ dynamical variable, $\underline{F}: \mathbb{R}^n \times \mathbb{R}_t \longrightarrow \mathbb{R}^n$ vectorfield

flow ϕ of vectorfield $\underline{F}$ on manifold M: $\phi: M \times \mathbb{R}_t \longrightarrow M$ (think $\mathbb{R}^n$)

with $\phi(\underline{x}_0, t) = \phi_t(\underline{x}_0) = \underline{x}(t; \underline{x}_0)$ (initial condition)

fixed point: $\dot{\underline{x}} = 0 \Rightarrow \underline{F}(\underline{x}^*) = 0$

of $\dot{\underline{x}} = \underline{F}(\underline{x})$

linear stability analysis: $\delta\dot{\underline{x}} = \frac{d}{dt}(\underline{x} - \underline{x}^*) = \left(\underline{\underline{DF}}\right)\Big|_{\underline{x}^*} \delta\underline{x}$ (Jacobian matrix)

ansatz: $\delta\underline{x} = \sum_{k=1}^{n} c_k \underline{\xi}^{(k)} e^{\lambda_k t}$ with $\lambda_k \underline{\xi}^{(k)} = \underline{\underline{DF}}\Big|_{\underline{x}^*} \underline{\xi}^{(k)}$ $k = 1, \dots, n$

eigenvalue eigenvector

example: SIR model with demography (Susceptible, Infected, Recovered)

$$\dot{S} = \mu - \beta S I - \mu S \qquad \beta: \text{infection rate}$$

$$\dot{I} = \beta S I - \gamma I - \mu I \qquad \gamma: \text{recovery rate}$$

$$\dot{R} = \gamma I - \mu R \qquad \mu: \text{birth/death rate}$$

$S+I+R=1$, $S, I, R \geq 0$

case 1: $\mu = 0 \Rightarrow I^* = 0 \Rightarrow S, R = \text{constant}$

$S(0) < \frac{\gamma}{\beta}$ disease dies out fast

$S(0) > \frac{\gamma}{\beta}$ outbreak

$R_0 = \frac{\beta}{\gamma}$: basic reproduction number/rate

case 2: $\mu > 0$: redefine $R_0 = \dfrac{\beta}{\gamma+\mu}$

fixed points: (i) $I^* = 0$: disease free (see above)

(ii) $S^* = \dfrac{1}{R_0} = \dfrac{\gamma+\mu}{\beta} \Rightarrow I^* = \dfrac{\mu}{\beta}(R_0 - 1) \Rightarrow R^* = \dfrac{\gamma}{\beta}(R_0 - 1)$

endemic state

linear stability analysis:

$$\underline{\underline{DF}}\Big|_{\underline{x}^*} = \begin{pmatrix} -\beta I^* - \mu & -\beta S^* & 0 \\ \beta I^* & \beta S^* - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

1.1 Dynamische Systeme (Fortsetzung)

Berechnen und lösen der charakteristischen Gleichung zur Bestimmung der Eigenwerte der Jacobi-Matrix:

$$0 \overset{!}{=} \det\left(\underline{\underline{DF}}|_{\underline{x}^*} - \Lambda \mathbb{1}\right) = (-\beta I^* - \mu - \Lambda)(\beta S^* - (\mu+\gamma) - \Lambda)(-\mu-\Lambda) - (\beta I^*)(-\beta S^*)(-\mu-\Lambda)$$

$$= \underbrace{(-\mu-\Lambda)}\left[(-\beta I^* - \mu - \Lambda)(\beta S^* - (\mu+\gamma) - \Lambda) + \beta I^* \beta S^*\right]$$

$\Rightarrow \Lambda_1 = -\mu < 0$ (stabile Richtung)

$\Lambda_{2,3}$ durch Lösen der quadratischen Gleichung $[\ldots] = 0$

Fall (i) $I^* = 0$, $S^* = 1$ (krankheitsfreier Fixpunkt)

$$\Rightarrow 0 = (-\mu-\Lambda)(\beta - (\gamma+\mu) - \Lambda)$$

$$\Rightarrow \Lambda_2 = -\mu < 0, \quad \Lambda_3 = \beta - (\gamma+\mu)$$

Fixpunkt ist stabil für $\Lambda_3 < 0 \Rightarrow \beta < \gamma + \mu \Leftrightarrow \frac{\beta}{\gamma+\mu} = R_0 < 1$

(ii) $S^* = \frac{1}{R_0}$, $I^* = \frac{\mu}{\beta}(R_0 - 1)$, $R^* = \frac{\gamma}{\beta}(R_0 - 1)$ (endemische Fixpunkt)

$$0 = [\ldots] = \ldots = \Lambda^2 + \mu R_0 \Lambda + \mu (R_0 - 1)(\gamma + \mu)$$

$$\Rightarrow \Lambda_{2,3} = -\frac{\mu R_0}{2} \pm \frac{\sqrt{(\mu R_0)^2 - \frac{4}{AG}}}{2} \quad \text{mit} \quad A = \frac{1}{\mu (R_0 - 1)}, \quad G = \frac{1}{\mu + \gamma}$$

A: Durchschnittsalter bei Ansteckung

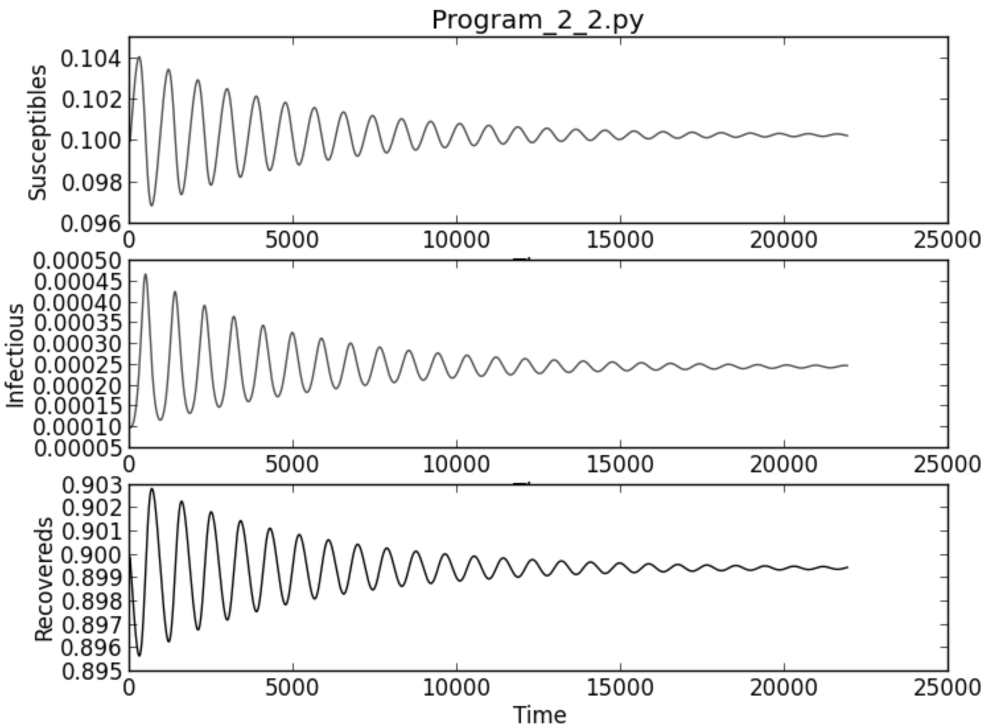
$$\underbrace{\frac{1}{\beta I^*}}_{\text{inverse Infektionskraft}} = \frac{1}{\mu (R_0 - 1)} = A \Rightarrow R_0 - 1 = \frac{1}{\mu A} = \underbrace{\frac{\text{Lebensdauer}}{A}}_{\Rightarrow \text{ siehe Tabelle } R_0 \text{ aus VL 2}}$$

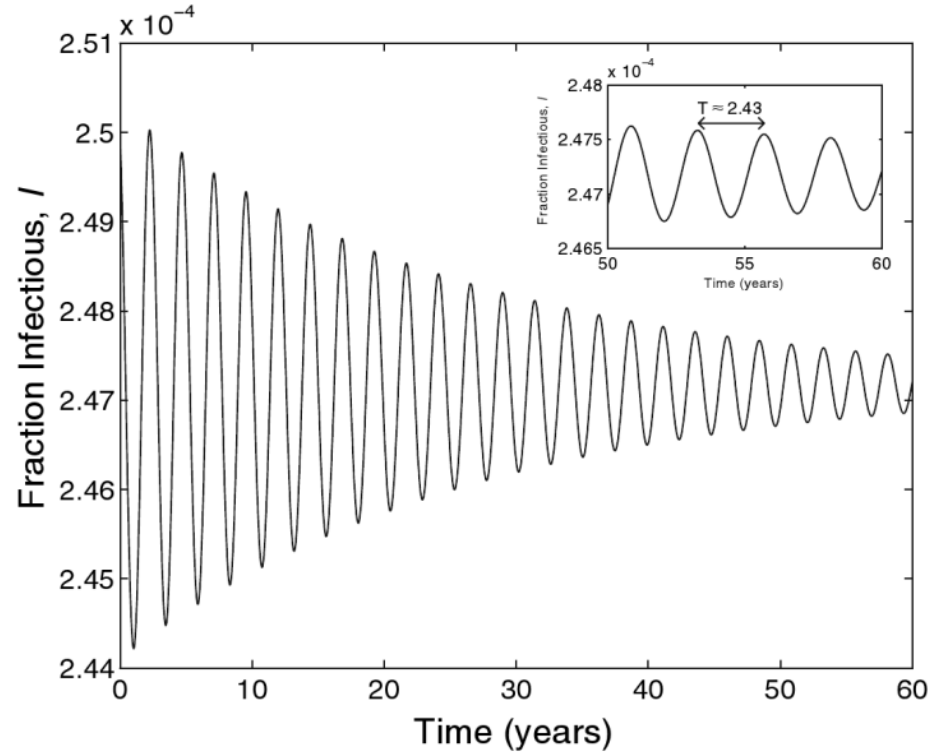
$$\Lambda_1 = -\mu < 0, \quad \Lambda_{2,3} = -\frac{\mu R_0}{2} \pm \frac{\sqrt{(\mu R_0)^2 - 4/AG}}{2}$$

$$\overset{(\mu R_0) \text{ klein}}{\approx} -\frac{\mu R_0}{2} \pm \frac{i}{\sqrt{AG}}$$

$\Rightarrow \mathrm{Re}(\Lambda_{2,3}) < 0 \Rightarrow$ endemischer Fixpunkt ist stabil

$$I(\Lambda_{2,3}) \approx \frac{1}{\sqrt{A\sigma}} \Rightarrow T = 2\pi\sqrt{A\sigma} = \frac{2\pi}{\sqrt{\mu(R_0-1)(\gamma+\mu)}}$$





$\frac{1}{\mu} = 70a$, $\beta = 520/a$, $\frac{1}{\gamma} = 7d \Rightarrow R_0 \simeq 10$, $I(0) = 2.5\cdot 10^{-4}$, $S(0) = 0.1$

$\Rightarrow T = 2.43a$

1.2 Einführung zu Netzwerken

Bsp.: physikal. / technisch: Verkehr, Internet, Strom, Transport ...

biologisches Gehirn, Handel, Klima ...

sozial: Facebook, Freundschaften, Koautorenschaft ...

⇒ Netzwerke sind überall

Dynamik auf Netzwerken ⟷ Dynamik von Netzwerken

	auf	von
NW-Struktur	fest	variable
Knotendynamik	variable	fest
#Knoten/Links	fest	ggf. veränderlich
Funktionalität	Knotenzustand	NW-Struktur

kombinierbar durch Wechselwirkungen: Rückkopplung der Dynamik auf NW-Struktur

⇒ adaptive Netzwerke

Einführende Literatur: • M. E. J. Newman: Networks: An Introduction

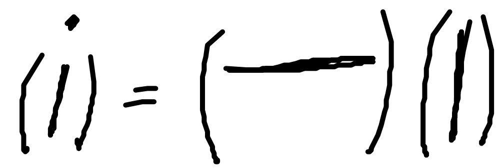
• A.-L. Barabasi: Network science

barabasi.com/networksciencebook

• VL-Folien barabasilab.neu.edu/courses/phys5116

Adjazenzmatrix: $\underline{\underline{A}} = \{a_{ij}\}_{i,j=1,\dots,N}$ mit $a_{ij} = \begin{cases} 1 & \text{wenn ein Link von Knoten } j \text{ nach } i \text{ existiert} \\ 0 & \text{sonst} \end{cases}$

Notation aus linearer Algebra (Matrix-Vektor-Multiplikation)



Achtung: Graphentheorie verwendet Notation $a_{ij} = 1$, wenn Link $i \to j$ existiert

(Transformation/Umrechnung durch Transposition von $\underline{\underline{A}}$)

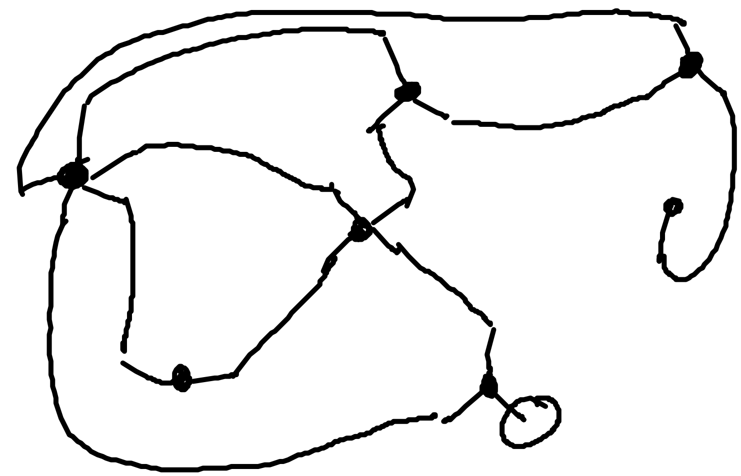
Netzwerkmaße: Eingangsgrad eines Knotens i: $k_i^{(in)} = \sum_{j=1}^{N} a_{ij}$

analog: Ausgangsgrad eines Knotens i: $k_i^{(out)} = \sum_{j=1}^{N} a_{ji}$

Konstruktion eines Netzwerks mit vorgegebener Gradverteilung:

(configuration model)

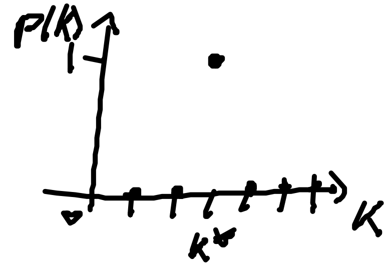
k	1	2	3	4	5
$p(k)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$



1. Rümpfe von $N=7$ Knoten mit Halblinks

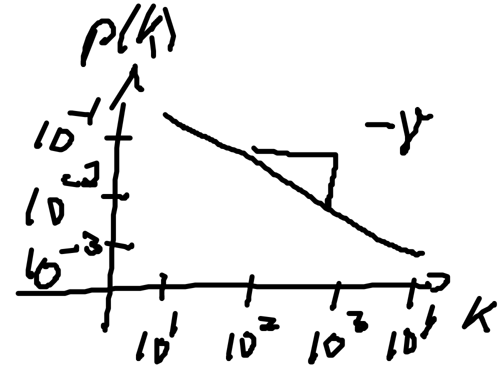
Summe aller Grade sollte gerade sein!

reguläres Netzwerk: alle Knoten haben den gleichen Grad k^*: $p(k) = \delta_{k,k^*}$



skalenfreies Netzwerk: $p(k) \sim k^{-\gamma}$

$\log p(k) \sim -\gamma \log k$



$\Rightarrow \log p(k) \sim \log k^{-\gamma}$

$\Rightarrow p(k) \sim k^{-\gamma}$

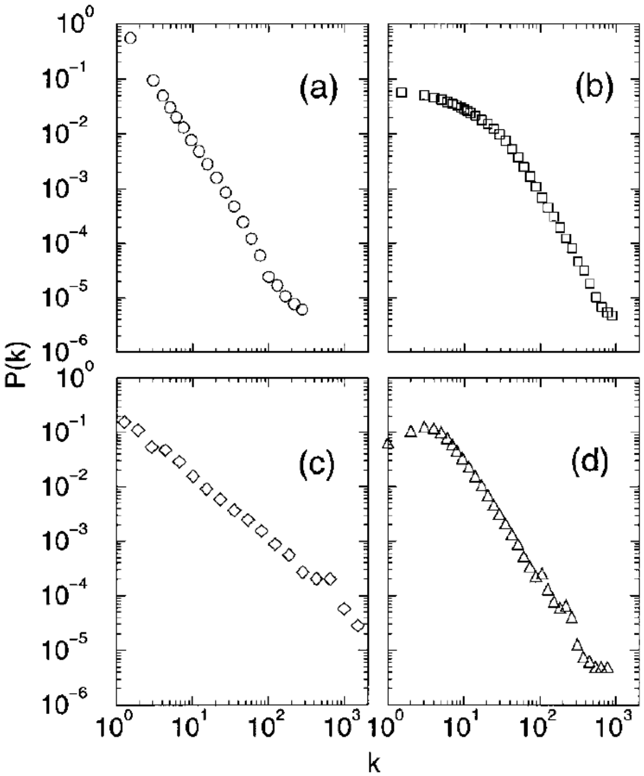


FIG. 3. The degree distribution of several real networks: (a) Internet at the router level. Data courtesy of Ramesh Govindan; (b) movie actor collaboration network. After Barabási and Albert 1999. Note that if TV series are included as well, which aggregate a large number of actors, an exponential cutoff emerges for large k (Amaral *et al.*, 2000); (c) co-authorship network of high-energy physicists. After Newman (2001a, 2001b); (d) co-authorship network of neuroscientists. After Barabási *et al.* (2001).

TABLE II. The scaling exponents characterizing the degree distribution of several scale-free networks, for which $P(k)$ follows a power law (2). We indicate the size of the network, its average degree $\langle k \rangle$, and the cutoff κ for the power-law scaling. For directed networks we list separately the indegree (γ_{in}) and outdegree (γ_{out}) exponents, while for the undirected networks, marked with an asterisk (*), these values are identical. The columns l_{real}, l_{rand}, and l_{pow} compare the average path lengths of real networks with power-law degree distribution and the predictions of random-graph theory (17) and of Newman, Strogatz, and Watts (2001) [also see Eq. (63) above], as discussed in Sec. V. The numbers in the last column are keyed to the symbols in Figs. 8 and 9.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference	N
WWW	325 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, and Barabási 1999	1
WWW	4×10^7	7		2.38	2.1				Kumar *et al.*, 1999	2
WWW	2×10^8	7.5	4000	2.72	2.1	16	8.85	7.61	Broder *et al.*, 2000	3
WWW, site	260 000				1.94				Huberman and Adamic, 2000	4
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2	4	6.3	5.2	Faloutsos, 1999	5
Internet, router*	3888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos, 1999	6
Internet, router*	150 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan, 2000	7
Movie actors*	212 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási and Albert, 1999	8
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2	4	2.12	1.95	Newman, 2001b	9
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.*, 2001	10
Co-authors, math.*	70 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.*, 2001	11
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.*, 2001	12
Metabolic, *E. coli*	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.*, 2000	13
Protein, *S. cerev.**	1870	2.39		2.4	2.4				Jeong, Mason, *et al.*, 2001	14
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya and Solé, 2000	14
Silwood Park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya and Solé, 2000	16
Citation	783 339	8.57			3				Redner, 1998	17
Phone call	53×10^6	3.16		2.1	2.1				Aiello *et al.*, 2000	18
Words, co-occurrence*	460 902	70.13		2.7	2.7				Ferrer i Cancho and Solé, 2001	19
Words, synonyms*	22 311	13.48		2.8	2.8				Yook *et al.*, 2001b	20

Exponenten γ liegen alle im Bereich um 2

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002

Statistical mechanics of complex networks

Réka Albert* and Albert-László Barabási

Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556

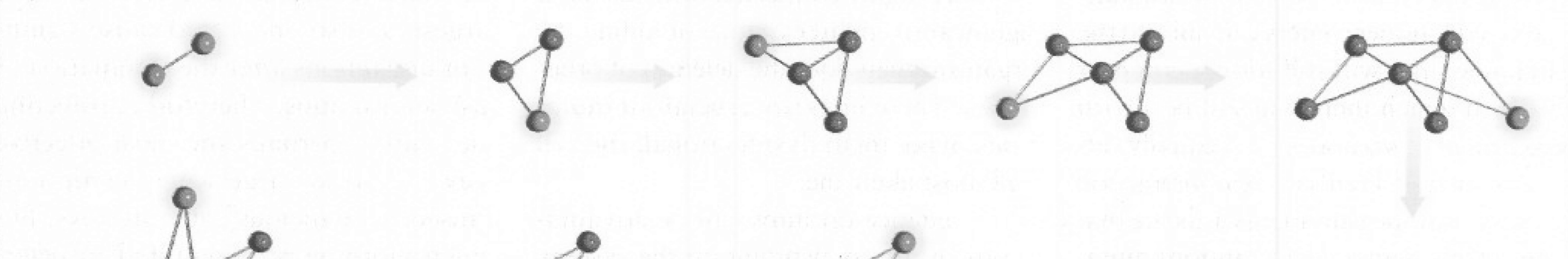
Idee für Konstruktion eines skalenfreien Netzwerks:

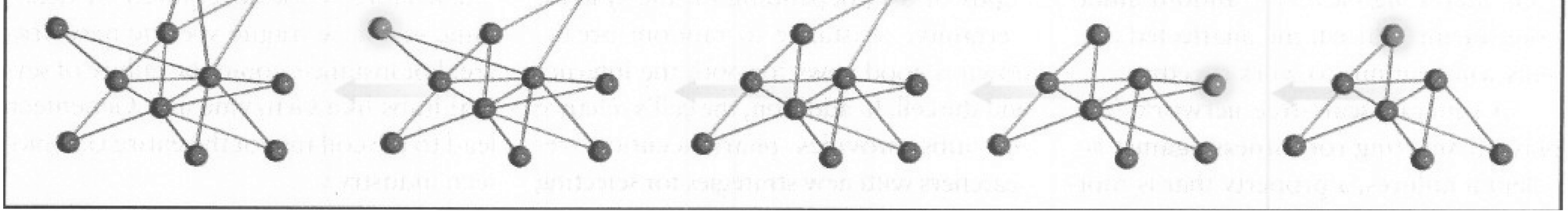
1) Wachstum: Knoten für Knoten einfügen mit m Links (start mit m_0) $m \leq m_0$

2) bevorzugtes Anlegen von Links mit Wahrscheinlichkeit $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$
preferential attachment

BIRTH OF A SCALE-FREE NETWORK

A SCALE-FREE NETWORK grows incrementally from two to 11 nodes in this example. When deciding where to establish a link, a new node (*green*) prefers to attach to an existing node (*red*) that already has many other connections. These two basic mechanisms—growth and preferential attachment—will eventually lead to the system's being dominated by hubs, nodes having an enormous number of links.





Frage: Strategien zum Entfernen der Knoten, sodass/bis das Netzwerk zerfällt?