Oscillator synchronization with common noise

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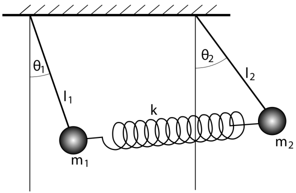


Overview

Synchronization in nature

Synchronization of coupled oscillators plays an important role in many different areas in nature, e.g. in

- the circadian clock (Hall, Rosbash, Young: Nobel prize 2017)
- neuronal networks
- swarms of birds, fish, and fireflies

Robin Meier & Andre Gwerder: Synchronicity (Thailand), 2015

Reasons for synchronization

Synchronization can be caused by different mechanisms:

- coupling
  - attractive $\rightarrow$ synchronization
  - repulsive $\rightarrow$ desynchronization
- (weak) common noise $\rightarrow$ synchronization

Interplay of repulsive coupling and common noise
$\rightarrow$ non-trivial effects

The Kuramoto Model

Infinitely many phase oscillators + Kuramoto-Sakaguchi coupling + common noise
discrete representation of phase-dynamics:

$$\dot{\varphi}_i = \Omega_i + \mu \sum_{\substack{j=1 \\ j\neq i}}^{N} \sin(\varphi_j - \varphi_i - \beta) + \sigma\xi \sin(\varphi_i)$$

Ω_i	natural frequency with distribution $g(\Omega) = \frac{\gamma}{\pi\left(\gamma^2+(\Omega-\Omega_0)^2\right)}$
μ	coupling
β	phase frustration parameter
$\xi(t)$	gaussian white noise with strength σ

Model Reduction

- $N \to \infty$: introduce probability density function $\omega(\varphi, t, \Omega)$ following $\frac{\partial \omega}{\partial t} + \frac{\partial(\omega \cdot v)}{\partial \varphi} = 0$

reduce the model according to the **Ott-Antonsen ansatz**:

- introduction of complex order parameter $z(t)$, $|z(t)| \leq 1$:

$$z(t) = Re^{i\Phi} = \int_{-\infty}^{\infty} \mathrm{d}\Omega g(\Omega) \int_0^{2\pi} d\varphi e^{i\varphi} \omega(\varphi, t, \Omega),$$

- describe ω by its Fourier series, using $\omega_n(\Omega, t) = a(\Omega, t)^n$:

$$\omega(\varphi, t, \Omega) = \frac{g(\Omega)}{2\pi} \left(1 + \sum_{n=1}^{\infty} \left(a(\Omega, t)^n \, e^{in\varphi} + a^*(\Omega, t)^n \, e^{-in\varphi} \right) \right)$$

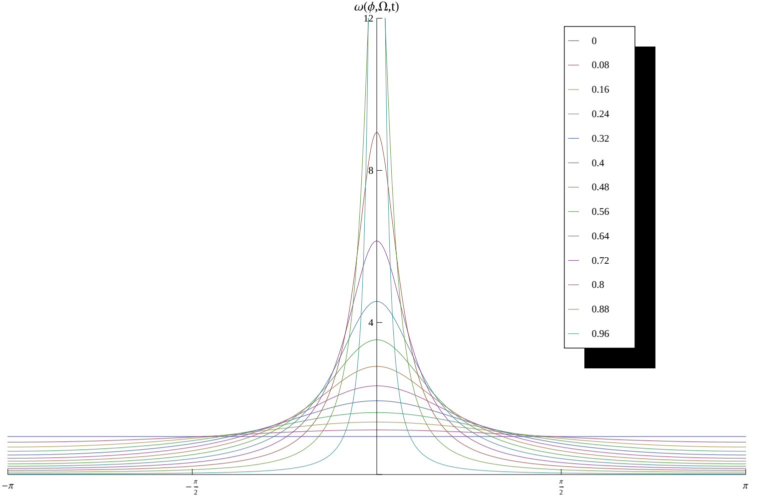


Figure: probability density function $\omega(\varphi, t, \Omega)$ as a function of φ (x-axis) and of a (→colored lines), which ranges from 0 (→horizontal line) to 1 (peak).

Reduced Equations

- deduce a special condition for $a(\Omega, t)$:

$$\dot{a}\,(\Omega, t) = -i\Omega\, a + \frac{\sigma\xi}{2}\left(a^2 - 1\right) + \frac{\mu}{2}\left(z^*\, e^{i\beta} - a^2\, z\, e^{-i\beta}\right) \;,$$

  that can be inserted into z(t).

- finally one obtains following equation of motion:

$$\begin{aligned}\dot{\varphi} &= \Omega + \sigma\xi(t)\sin(\varphi) + \mu R\sin(\Phi - \varphi - \beta)\\ &= \Omega + \operatorname{Im}\left(H(t)e^{-i\varphi}\right) \;,\end{aligned}$$

  with $H(t) = \mu R e^{-i\beta} e^{i\Phi} - \sigma\xi(t)$

Dynamics of the order parameter

Transformation of the order parameter $R \to J = R^2 / \left(1 - R^2\right)$

$$R \in [0,1], \quad J \in [\,0, \infty\,[$$

where $R = 0 / J = 0 \Leftrightarrow$ asynchrony, $\quad R = 1 / J \to \infty \Leftrightarrow$ synchrony

$$\dot{J} = \mu_\beta J - 2\gamma J(J+1) - \sigma\xi(t)\sqrt{J(J+1)}\cos(\Phi)$$

$$\dot{\Phi} = \Omega_0 - \mu\sin(\beta)\frac{J+1/2}{J+1} + \sigma\xi(t)\frac{J+1/2}{\sqrt{J(J+1)}}\sin(\Phi)$$

Stability of the synchronous state for $J \gg 1$

$$\dot{J} = \mu_\beta J - 2\gamma J^2 - \sigma\xi(t) J \cos(\Phi)$$
$$\dot{\Phi} = \Omega_{\mu,0} + \sigma\xi(t)\sin(\Phi)$$

Average growth rate of J for $\gamma = 0$ (identical oscillators):

$$\lambda_0 = \underbrace{\mu_\beta}_{\lesseqgtr 0} + \underbrace{\sigma^2 \left\langle \sin^2(\Phi) \right\rangle}_{>0} \Rightarrow \text{perfect synchrony possible}$$

Average growth rate of J for $\gamma \neq 0$ (non-identical oscillators):

$$\lambda_\gamma = \mu_\beta - 2\gamma \left\langle J \right\rangle + \sigma^2 \left\langle \sin^2(\Phi) \right\rangle \Rightarrow \text{perfect synchrony impossible}$$

Results

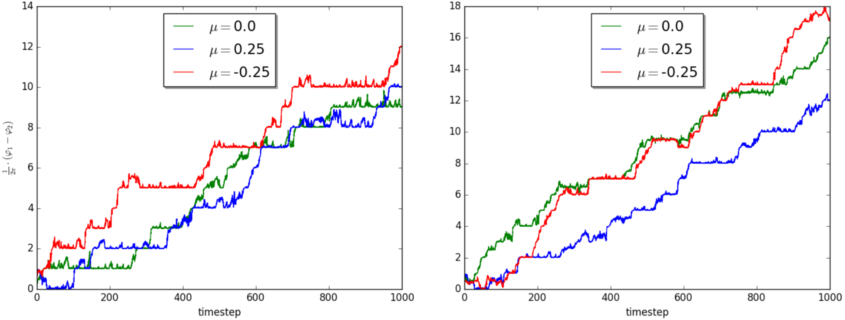
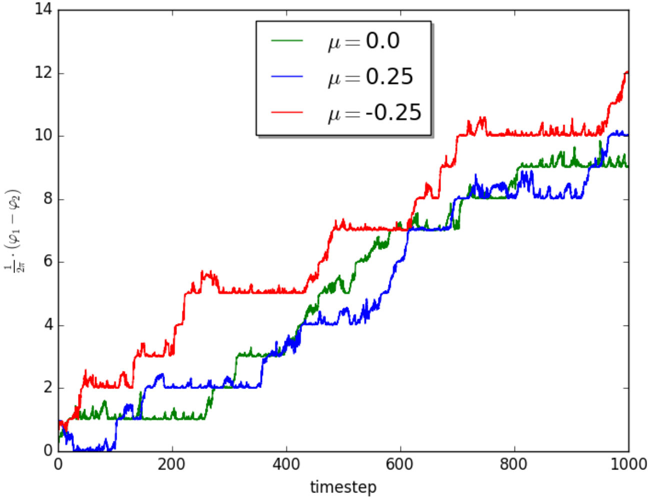


Figure: Dynamics of the frequency difference $\frac{\varphi_i - \varphi_j}{2\pi}$ of two oscillators i, j in an ensemble for attractive ($\mu = 0.25$), repulsive ($\mu = -0.25$) or no coupling ($\mu = 0$).

Results


14
12
10
8
6
4
2
0
0
200
400
600
800
1000
timestep
$\frac{1}{2\pi} \cdot (\varphi_1 - \varphi_2)$
$\mu = 0.0$
$\mu = 0.25$
$\mu = -0.25$

Influence of coupling and noise on the synchronization of non-identical oscillators

- coupling without noise:
  - attraction of frequencies for positive coupling ($\mu > 0$)
  - no effect for repulsive coupling ($\mu < 0$)
- noise without coupling: no influence on the frequencies
- coupling together with noise:
  - $\mu > 0$: same as without noise
  - $\mu < 0$: dispersion of frequencies

  $\rightarrow$ **counterintuitive !**

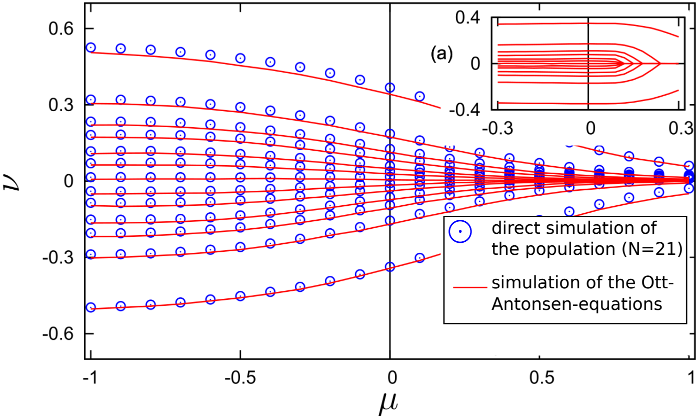


Figure: adapted from Figure 3. in : A. V. Pimenova, D.S. Goldobin, M. Rosenblum and A. Pikovsky, *Interplay of coupling and common noise at the transition to synchrony in oscillator populations*, Scientific Reports 38518 (2016)

Chemical oscillators

Mixture of reacting chemical compounds, in which the **concentration** of one or more components exhibits periodic changes

- non-equillibrium thermodynamics
- state of chemical oscillators described by intensity I
- intensity depends on concentration
  $\rightarrow$influences grey value g of each oscillator

Coupled chemical oscillators

- Setup:

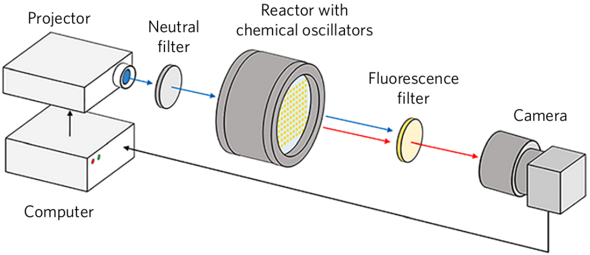
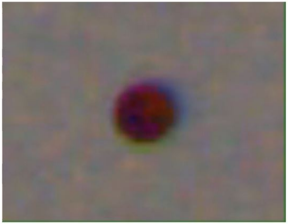


Figure: adapted: Fig. 1. in : J.F. Totz, J. Rode, M.R. Tinsley, K. Showalter, H. Engel, *Spiral wave chimera states in large populations of coupled chemical oscillators*, Nature Physics Vol.14, 282-5, 2018

- population of asynchronous chemical oscillators
- placed in a catalyst-free Belousov-Zhabotinsky (BZ) solution
- coupling: through light, non-local

Coupled chemical oscillators



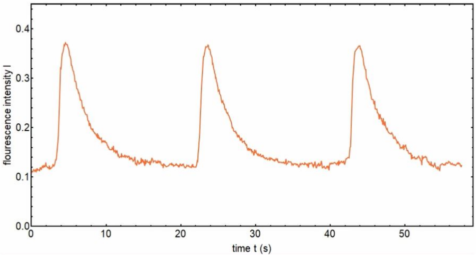


Figure: adapted from: J.F. Totz, J. Rode, M.R. Tinsley, K. Showalter and H. Engel, *Spiral wave chimera states in large populations of coupled chemical oscillators*, Nature Physics Vol.14, 282-285 (2018)

Model

- oscillators (at position (j,k)) arranged on a 2D-grid:

$$I_{j,k} = I_0 + K \sum_{m=j-l}^{j+l} \sum_{n=k-l}^{k+l} e^{-\kappa r} \left(g_{m,n}(t-\tau) - g_{j,k}(t) \right)$$

K	coupling strength, κ coupling range
$e^{-\kappa r}$	non-local coupling kernel, $r = \sqrt{(m-j)^2 + (n-k)^2}$
I_0	intensity of the background illumination
τ	time-delay $\Leftrightarrow$ phase-frustration parameter

Implementation of noise

similarly to presented model in Ref.[1]:

- global coupling
- multiplicative noise

$$I_i = I_0 + \mu \sum_{j=1}^{N} (g_j(t) - g_i(t)) + \sigma \xi(t) f(g_i)$$

μ	coupling strength	σ	noise intensity
$\xi(t)$	gaussian noise	$f(g_i)$	function of grey value g_i

[1] S. Goldobin, A. V. Pimenova, M. Rosenblum and A. Pikovsky, *Competing influence of common noise and desynchronizing coupling on synchronization in the Kuramoto-Sakaguchi ensemble*, Eur. Phys. J. Special Topics 226, 1921-1937 (2017)

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- D.S. Goldobin, A. V. Pimenova, M. Rosenblum and A. Pikovsky, *Interplay of coupling and common noise at the transition to synchrony in oscillator populations*, Nature Scientific Reports 38518 (2016)

J.F. Totz, J. Rode, M.R. Tinsley, K. Showalter and H. Engel, *Spiral wave chimera states in large populations of coupled chemical oscillators*, Nature Physics Vol.14, 282-285 (2018)

J. F. Totz., *Synchronization and Waves in Confined Complex Active Media*, Ph.D. thesis, TU Berlin, Berlin (2017)

Thank you for your attention!